MECHANICS

by

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5-8 Stress and strain. If an imaginary surface cuts through any part of a solid structure (a rod, string, cable, or beam), then, in general, the material on one side of this surface will be exerting a force on the material
on the other side, and conversely, according to Newton's third law. These internal forces which act across any surface within the solid are called stresses. The stress is defined as the force per unit area acting across any given surface in the material. If the material on each side of any surface pushes on the material on the other side with a force perpendicular to the surface, the stress is called a compression. If the stress is a pull perpendicular to the surface, it is called a tension. If the force exerted across the surface is parallel to the surface, it is called a shearing stress. Figure 5–20 illustrates these stresses in the case of a beam. The vector labeled $F_{r \rightarrow r}$ represents the force exerted by the left half of the beam on the right half, and the equal and opposite force $F_{r \rightarrow l}$ is exerted on the material on the left by the material on the right. A stress at an angle to a surface can be resolved into a shear component and a tension or compression component.

In the most general case, the stress may act in any direction relative to the surface, and may depend on the orientation of the surface. The description of the state of stress of a solid material in the most general case is rather complicated, and is best accomplished by using the mathematical techniques of tensor algebra to be developed in Chapter 10. We shall consider here only cases in which either the stress is a pure compression, independent of the orientation of the surface, or in which only one surface is of interest at any point, so that only a single stress vector is needed to specify the force per unit area across that surface.

If we consider a small volume $\Delta V$ of any shape in a stressed material, the material within this volume will be acted on by stress forces exerted across the surface by the material surrounding it. If the material is not perfectly rigid, it will be deformed so that the material in the volume $\Delta V$ may have a different shape and size from that which it would have if there were no stress. This deformation of a stressed material is called strain.
The nature and amount of strain depend on the nature and magnitude of the stresses and on the nature of the material. A suitable definition of strain, stating how it is to be measured, will have to be made for each kind of strain. A tension, for example, produces an extension of the material, and the strain would be defined as the fractional increase in length.

If a wire of length \( l \) and cross-sectional area \( A \) is stretched to length \( l + \Delta l \) by a force \( F \), the definitions of stress and strain are

\[
\text{stress} = \frac{F}{A}, \quad (5-112)
\]
\[
\text{strain} = \frac{\Delta l}{l}. \quad (5-113)
\]

It is found experimentally that when the strain is not too large, the stress is proportional to the strain for solid materials. This is Hooke’s law, and it is true for all kinds of stress and the corresponding strains. It is also plausible on theoretical grounds for the reasons suggested in the preliminary discussion in Section 2–7. The ratio of stress to strain is therefore constant for any given material if the strain is not too large. In the case of extension of a material in one direction due to tension, this ratio is called Young’s modulus, and is

\[
Y = \frac{\text{stress}}{\text{strain}} = \frac{Fl}{A \Delta l}. \quad (5-114)
\]

If a substance is subjected to a pressure increment \( \Delta p \), the resulting deformation will be a change in volume, and the strain will be defined by

\[
\text{strain} = \frac{\Delta V}{V}. \quad (5-115)
\]

The ratio of stress to strain in this case is called the bulk modulus \( B \):

\[
B = \frac{\text{stress}}{\text{strain}} = -\frac{\Delta p V}{\Delta V}. \quad (5-116)
\]

where the negative sign is introduced in order to make \( B \) positive.

In the case of a shearing stress, the stress is again defined by Eq. (5–112), where \( F \) is the force acting across and parallel to the area \( A \). The resulting shearing strain consists in a motion of \( A \) parallel to itself through a distance \( \Delta l \), relative to a plane parallel to \( A \) at a distance \( \Delta x \) from \( A \) (Fig. 5–21). The shearing strain is then defined by

\[
\text{strain} = \frac{\Delta l}{\Delta x} = \tan \theta, \quad (5-117)
\]

where \( \theta \) is the angle through which a line perpendicular to \( A \) is turned as a result of the shearing strain. The ratio of stress to strain in this case is
called the *shear modulus* \( n \):

\[
 n = \frac{\text{stress}}{\text{strain}} = \frac{F}{A \tan \theta}.
\]  

(5–118)

An extensive study of methods of solving problems in statics is outside the scope of this text. We shall restrict ourselves in the next three sections to the study of three special types of problems which illustrate the analysis of a physical system, to determine the forces which act upon its parts and to determine the effect of these forces in deforming the system.
5-11 Equilibrium of fluids. A fluid is defined as a substance which will support no shearing stress when in equilibrium. Liquids and gases fit this definition, and even very viscous substances like pitch, or tar, or the material in the interior of the earth, will eventually come to an equilibrium in which shearing stresses are absent, if they are left undisturbed for a sufficiently long time. The stress \( F/A \) across any small area \( A \) in a fluid in equilibrium must be normal to \( A \), and in practically all cases it will be a compression rather than a tension.

We first prove that the stress \( F/A \) near any point in the fluid is independent of the orientation of the surface \( A \). Let any two directions be given, and construct a small triangular prism with two equal faces \( A_1 = A_2 \) perpendicular to the two given directions. The third face \( A_3 \) is to form with \( A_1 \) and \( A_2 \) a cross section having the shape of an isosceles triangle (Fig. 5-27). Let \( F_1, F_2, F_3 \) be the stress forces perpendicular to the faces
Fig. 5-27. Forces on a triangular prism in a fluid.

$A_1, A_2, A_3$. If the fluid in the prism is in equilibrium,

$$F_1 + F_2 + F_3 = 0.$$  \hspace{1cm} (5-166)

The forces on the end faces of the prism need not be included here, since they are perpendicular to $F_1$, $F_2$, and $F_3$, and must therefore separately add to zero. It follows from Eq. (5-166), and from the way the prism has been constructed, that $F_1$, $F_2$, and $F_3$ must form an isosceles triangle (Fig. 5-27), and therefore that

$$F_1 = F_2.$$  \hspace{1cm} (5-167)

Since the directions of $F_1$ and $F_2$ are any two directions in the fluid, and since $A_1 = A_2$, the stress $F/A$ is the same in all directions. The stress in a fluid is called the pressure $p$:

$$p = \frac{F_1}{A_1} = \frac{F_2}{A_2}.$$  \hspace{1cm} (5-168)

Now suppose that in addition to the pressure the fluid is subject to an external force $f$ per unit volume of fluid, that is, any small volume $dV$ in the fluid is acted on by a force $f \, dV$. Such a force is called a body force; $f$ is the body force density. The most common example is the gravitational force, for which

$$f = \rho g,$$  \hspace{1cm} (5-169)

where $g$ is the acceleration of gravity, and $\rho$ is the density. In general, the body force density may differ in magnitude and direction at different points in the fluid. In the usual case, when the body force is given by Eq. (5-169), $g$ will be constant and $f$ will be constant in direction; if $\rho$ is constant, $f$ will also be constant in magnitude. Let us consider two nearby points $P_1, P_2$ in the fluid, separated by a vector $dr$. We construct a cylinder of length $dr$ and cross-sectional area $dA$, whose end faces contain the points $P_1$ and $P_2$. Then the total component of force in the direction of $dr$ acting
on the fluid in the cylinder, since the fluid is in equilibrium, will be

\[ f \cdot dr \, dA + p_1 \, dA - p_2 \, dA = 0, \]

where \( p_1 \) and \( p_2 \) are the pressures at \( P_1 \) and \( P_2 \). The difference in pressure between two points a distance \( dr \) apart is therefore

\[ dp = p_2 - p_1 = f \cdot dr. \]  \hspace{1cm} (5-170)

The total difference in pressure between two points in the fluid located by vectors \( r_1 \) and \( r_2 \) will be

\[ p_2 - p_1 = \int_{r_1}^{r_2} f \cdot dr, \]  \hspace{1cm} (5-171)

where the line integral on the right is to be taken along some path lying entirely within the fluid from \( r_1 \) to \( r_2 \). Given the pressure \( p_1 \) at \( r_1 \), Eq. (5-171) allows us to compute the pressure at any other point \( r_2 \) which can be joined to \( r_1 \) by a path lying within the fluid. The difference in pressure between any two points depends only on the body force. Hence any change in pressure at any point in a fluid in equilibrium must be accompanied by an equal change at all other points if the body force does not change. This is Pascal's law.

According to the geometrical definition (3-107) of the gradient, Eq. (5-170) implies that

\[ f = \nabla p. \]  \hspace{1cm} (5-172)

The pressure gradient in a fluid in equilibrium must be equal to the body force density. This result shows that the net force per unit volume due to pressure is \(-\nabla p\). The pressure \( p \) is a sort of potential energy per unit volume in the sense that its negative gradient represents a force per unit volume due to pressure. However, the integral of \( p \, dV \) over a volume does not represent a potential energy except in very special cases. Equation (5-172) implies that the surfaces of constant pressure in the fluid are everywhere perpendicular to the body force. According to Eqs. (3-187) and (5-172), the force density \( f \) must satisfy the equation

\[ \nabla \times f = 0. \]  \hspace{1cm} (5-173)

This is therefore a necessary condition on the body force in order for equilibrium to be possible. It is also a sufficient condition for the possibility of equilibrium. This follows from the discussion in Section 3-12, for if Eq. (5-173) holds, then it is permissible to define a function \( p(r) \) by the equation

\[ p(r) = p_1 + \int_{r_1}^{r} f \cdot dr, \]  \hspace{1cm} (5-174)
where \( p_1 \) is the pressure at some fixed point \( r_1 \), and the integral may be evaluated along any path from \( r_1 \) to \( r \) within the fluid. If the pressure in the fluid at every point \( r \) has the value \( p(r) \) given by (5-174), then Eq. (5-172) will hold, and the body force \( f \) per unit volume will everywhere be balanced by the pressure force \(-\nabla p\) per unit volume. Equation (5-174) therefore defines an equilibrium pressure distribution for any body force satisfying Eq. (5-173).

The problem of finding the pressure within a fluid in equilibrium, if the body force density \( f(r) \) is given, is evidently mathematically identical with the problem discussed in Section 3-12 of finding the potential energy for a given force function \( F(r) \). We first check that \( \nabla \times f \) is zero everywhere within the fluid, in order to be sure that an equilibrium is possible. We then take a point \( r_1 \) at which the pressure is known, and use Eq. (5-174) to find the pressure at any other point, taking the integral along any convenient path.

The total body force acting on a volume \( V \) of the fluid is

\[ F_b = \iiint_V f \, dV. \tag{5-175} \]

The total force due to the pressure on the surface \( A \) of \( V \) is

\[ F_p = -\iint_A np \, dA, \tag{5-176} \]

where \( n \) is the outward normal unit vector at any point on the surface. These two must be equal and opposite, since the fluid is in equilibrium:

\[ F_p = -F_b. \tag{5-177} \]

Equation (5-176) gives the total force due to pressure on the surface of the volume \( V \), whether or not \( V \) is occupied by fluid. Hence we conclude from Eq. (5-177) that a body immersed in a fluid in equilibrium is acted on by a force \( F_p \) due to pressure, equal and opposite to the body force \( F_b \), which would be exerted on the volume \( V \) if it were occupied by fluid in equilibrium. This is Archimedes' principle. Combining Eqs. (5-172), (5-175), (5-176), and (5-177), we have

\[ \iint_A np \, dA = \iiint_V \nabla p \, dV. \tag{5-178} \]

This equation resembles Gauss' divergence theorem [Eq. (3-115)], except that the integrands are \( np \) and \( \nabla p \) instead of \( n \cdot A \) and \( \nabla \cdot A \). Gauss' theorem can, in fact, be proved in a very useful general form which allows us to replace the factor \( n \) in a surface integral by \( \nabla \) in the corresponding volume
integral without any restrictions on the form of the integrand except that it must be so written that the differentiation symbol \( \nabla \) operates on the entire integrand.\(^*\) Given this result, we could start with Eqs. (5-175), (5-176), and (5-177), and deduce Eq. (5-172):

\[
\mathbf{F} + \mathbf{F}_p = \int \int \int f \, dV - \int \int \mathbf{n} p \, dA
\]

\[
= \int \int \int (f - \nabla p) \, dV = 0. \tag{5-179}
\]

Since this must hold for any volume \( V \), Eq. (5-172) follows.

So far we have been considering only the pressure, i.e., the stress, in a fluid. The strain produced by the pressure within a fluid is a change in volume per unit mass of the fluid or, equivalently, a change in density. If Hooke's law is satisfied, the change \( dV \) in a volume \( V \) produced by a small change \( dp \) in pressure can be calculated from Eq. (5-116), if the bulk modulus \( B \) is known:

\[
\frac{dV}{V} = -\frac{dp}{B}. \tag{5-180}
\]

If the mass of fluid in the volume \( V \) is \( M \), then the density is

\[
\rho = \frac{M}{V}, \tag{5-181}
\]

and the change \( dp \) in density corresponding to an infinitesimal change \( dV \) in volume is given by

\[
\frac{dp}{\rho} = -\frac{dV}{V}, \tag{5-182}
\]

so that the change in density produced by a small pressure change \( dp \) is

\[
\frac{dp}{\rho} = \frac{dp}{B}. \tag{5-183}
\]

After a finite change in pressure from \( p_0 \) to \( p \), the density will be

\[
\rho = \rho_0 \exp \left( \int_{p_0}^{p} \frac{dp}{B} \right). \tag{5-184}
\]

In any case, the density of a fluid is determined by its equation of state in

\(^*\) For the proof of this theorem, see Phillips, Vector Analysis. New York: John Wiley and Sons, 1933. (Chapter III, Section 34.)
terms of the pressure and temperature. The equation of state for a perfect gas is

\[ pV = RT, \quad (5-185) \]

where \( T \) is the absolute temperature, \( V \) is the volume per mole, and \( R \) is the universal gas constant:

\[ R = 8.314 \times 10^7 \text{ erg-deg}^{-1} \text{ C-mole}^{-1}. \quad (5-186) \]

By substitution from Eq. (5-181), we obtain the density in terms of pressure and temperature:

\[ \rho = \frac{M_p}{RT}, \quad (5-187) \]

where \( M \) is the molecular weight.

Let us apply these results to the most common case, in which the body force is the gravitational force on a fluid in a uniform vertical gravitational field [Eq. (5-169)]. If we apply Eq. (5-173) to this case, we have

\[ \nabla \times \mathbf{f} = \nabla \times (\rho g) = 0. \quad (5-188) \]

Since \( g \) is constant, the differentiation implied by the \( \nabla \) symbol operates only on \( \rho \), and we can move the scalar \( \rho \) from one factor of the cross product to the other to obtain:

\[ (\nabla \rho) \times g = 0, \quad (5-189) \]

that is, the density gradient must be parallel to the gravitational field. The density must be constant on any horizontal plane within the fluid. Equation (5-189) may also be derived from Eq. (5-188) by writing out explicitly the components of the vectors \( \nabla \times (\rho g) \) and \( (\nabla \rho) \times g \), and verifying that they are the same.* According to Eq. (5-172), the pressure is also constant in any horizontal plane within the fluid. Pressure and density are therefore functions only of the vertical height \( z \) within the fluid. From Eqs. (5-172) and (5-169) we obtain a differential equation for pressure as a function of \( z \):

\[ \frac{dp}{dz} = -\rho g. \quad (5-190) \]

If the fluid is incompressible, and \( \rho \) is uniform, the solution is

\[ p = p_0 - \rhogz, \quad (5-191) \]

---

* Equation (5-189) holds also in a nonuniform gravitational field, since \( \nabla \times g = 0 \), by Eq. (6-21).
where \( p_0 \) is the pressure at \( z = 0 \). If the fluid is a perfect gas, either \( p \) or \( \rho \) may be eliminated from Eq. (5-190) by means of Eq. (5-187). If we eliminate the density, we have

\[
\frac{dp}{dz} = -\frac{Mg}{RT} p. \tag{5-192}
\]

As an example, if we assume that the atmosphere is uniform in temperature and composition, we can solve Eq. (5-192) for the atmospheric pressure as a function of altitude:

\[
p = p_0 \exp \left( -\frac{Mg}{RT} z \right). \tag{5-193}
\]
8–6 Kinematics of moving fluids. In this section we shall develop the kinematic concepts useful in studying the motion of continuously distributed matter, with particular reference to moving fluids. One way of describing the motion of a fluid would be to attempt to follow the motion of each individual point in the fluid, by assigning coordinates \( x, y, z \) to each fluid particle and specifying these as functions of the time. We may, for example, specify a given fluid particle by its coordinates, \( x_0, y_0, z_0 \), at an initial instant \( t = t_0 \). We can then describe the motion of the fluid by means of functions \( x(x_0, y_0, z_0, t), y(x_0, y_0, z_0, t), z(x_0, y_0, z_0, t) \) which determine the coordinates \( x, y, z \) at time \( t \) of the fluid particle which was at \( x_0, y_0, z_0 \) at time \( t_0 \). This would be an immediate generalization of the concepts of particle mechanics, and of the preceding treatment of the vibrating string. This program originally due to Euler leads to the so-called “Lagrangian equations” of fluid mechanics. A more convenient treatment for many purposes, due also to Euler, is to abandon the attempt to specify the history of each fluid particle, and to specify instead the density and velocity of the fluid at each point in space at each instant of time. This is the method which we shall follow here. It leads to the “Eulerian equations” of fluid mechanics. We describe the motion of the fluid by specifying the density \( \rho(x, y, z, t) \) and the vector velocity \( \mathbf{v}(x, y, z, t) \), at the point \( x, y, z \) at the time \( t \). We thus focus our attention on what is happening at a particular point in space at a particular time, rather than on what is happening to a particular fluid particle.
Any quantity which is used in describing the state of the fluid, for example the pressure $p$, will be a function $[p(x, y, z, t)]$ of the space coordinates $x$, $y$, $z$ and of the time $t$; that is, it will have a definite value at each point in space and at each instant of time. Although the mode of description we have adopted focuses attention on a point in space rather than on a fluid particle, we shall not be able to avoid following the fluid particles themselves, at least for short time intervals $dt$. For it is to the particles, and not to the space points, that the laws of mechanics apply. We shall be interested, therefore, in two time rates of change for any quantity, say $p$. The rate at which the pressure is changing with time at a fixed point in space will be the partial derivative with respect to time ($\partial p/\partial t$); it is itself a function of $x$, $y$, $z$, and $t$. The rate at which the pressure is changing with respect to a point moving along with the fluid will be the total derivative

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + \frac{\partial p}{\partial x} \frac{dx}{dt} + \frac{\partial p}{\partial y} \frac{dy}{dt} + \frac{\partial p}{\partial z} \frac{dz}{dt},$$

(8–110)

where $dx/dt$, $dy/dt$, $dz/dt$ are the components of the fluid velocity $v$. The change in pressure, $dp$, occurring during a time $dt$, at the position of a moving fluid particle which moves from $x$, $y$, $z$ to $x + dx$, $y + dy$, $z + dz$ during this time, will be

$$dp = p(x + dx, y + dy, z + dz, t + dt) - p(x, y, z, t)$$

$$= \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz + \frac{\partial p}{\partial t} dt,$$

and if $dt \to 0$, this leads to Eq. (8–110). We can also write Eq. (8–110) in the forms:

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + v_x \frac{\partial p}{\partial x} + v_y \frac{\partial p}{\partial y} + v_z \frac{\partial p}{\partial z}$$

(8–111)

and

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{v} p,$$

(8–112)

where the second expression is a shorthand for the first, in accordance with the conventions for using the symbol $\nabla$. The total derivative $dp/dt$ is also a function of $x$, $y$, $z$, and $t$. A similar relation holds between partial and total derivatives of any quantity, and we may write, symbolically,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla,$$

(8–113)

where total and partial derivatives have the meaning defined above.

Let us consider now a small volume $\delta V$ of fluid, and we shall agree that $\delta V$ always designates a volume element which moves with the fluid, so
that it always contains the same fluid particles. In general, the volume \( \delta V \) will then change with time, and we wish to calculate this rate of change. Let us assume that \( \delta V \) is in the form of a rectangular box of dimensions \( \delta x, \delta y, \delta z \) (Fig. 8-4):

\[
\delta V = \delta x \delta y \delta z. \tag{8-114}
\]

The \( x \)-component of fluid velocity \( v_x \) may be different at the left and right faces of the box. If so, \( \delta x \) will change with time at a rate equal to the difference between these two velocities:

\[
\frac{d}{dt} \delta x = \frac{\partial v_x}{\partial x} \delta x,
\]

and, similarly,

\[
\begin{cases}
\frac{d}{dt} \delta y = \frac{\partial v_y}{\partial y} \delta y, \\
\frac{d}{dt} \delta z = \frac{\partial v_z}{\partial z} \delta z.
\end{cases} \tag{8-115}
\]

The time rate of change of \( \delta V \) is then

\[
\frac{d}{dt} \delta V = \delta y \delta z \frac{d}{dt} \delta x + \delta x \delta z \frac{d}{dt} \delta y + \delta x \delta y \frac{d}{dt} \delta z
\]

\[
= \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \delta x \delta y \delta z,
\]

and finally,

\[
\frac{d}{dt} \delta V = \nabla \cdot \mathbf{v} \delta V. \tag{8-116}
\]

This derivation is not very rigorous, but it gives an insight into the meaning of the divergence \( \nabla \cdot \mathbf{v} \). The derivation can be made rigorous by keeping careful track of quantities that were neglected here, like the de-
dependence of \( v_x \) upon \( y \) and \( z \), and showing that we arrive at Eq. (8-116) in the limit as \( \delta V \to 0 \). However, there is an easier way to give a more rigorous proof of Eq. (8-116). Let us consider a volume \( V \) of fluid which is composed of a number of elements \( \delta V \):

\[
V = \sum \delta V. \tag{8-117}
\]

If we sum the left side of Eq. (8-116), we have

\[
\sum \frac{d}{dt} \delta V = \frac{d}{dt} \sum \delta V = \frac{dV}{dt}. \tag{8-118}
\]

The summation sign here really represents an integration, since we mean to pass to the limit \( \delta V \to 0 \), but the algebraic steps in Eq. (8-118) would look rather unfamiliar if the integral sign were used. Now let us sum the right side of Eq. (8-116), this time passing to the limit and using the integral sign, in order that we may apply Gauss' divergence theorem [Eq. (3-115)]:

\[
\sum \nabla \cdot \mathbf{v} \delta V = \int \int \int_{V} \nabla \cdot \mathbf{v} \, dV = \int \int_{S} \mathbf{n} \cdot \mathbf{v} \, dS, \tag{8-119}
\]

where \( S \) is the surface bounding the volume \( V \), and \( \mathbf{n} \) is the outward normal unit vector. Since \( \mathbf{n} \cdot \mathbf{v} \) is the outward component of velocity of the surface element \( dS \), the volume added to \( V \) by the motion of \( dS \) in a time \( dt \) will be \( \mathbf{n} \cdot \mathbf{v} \, dt \, dS \) (Fig. 8-5), and hence the last line in Eq. (8-119) is the proper expression for the rate of increase in volume:

\[
\frac{dV}{dt} = \int \int_{S} \mathbf{n} \cdot \mathbf{v} \, dS. \tag{8-120}
\]

Therefore Eq. (8-116) must be the correct expression for the rate of

Fig. 8-5. Increase of volume due to motion of surface.
increase of a volume element, since it gives the correct expression for the rate of increase of any volume \( V \) when summed over \( V \). Note that the proof is independent of the shape of \( \delta V \). We have incidentally derived an expression for the time rate of change of a volume \( V \) of moving fluid:

\[
\frac{dV}{dt} = \iiint_V \nabla \cdot \mathbf{v} \, dV. \tag{8-121}
\]

If the fluid is incompressible, then the volume of every element of fluid must remain constant:

\[
\frac{d}{dt} \delta V = 0, \tag{8-122}
\]

and consequently, by Eq. (8-116),

\[
\nabla \cdot \mathbf{v} = 0. \tag{8-123}
\]

No fluid is absolutely incompressible, but for many purposes liquids may be regarded as practically so and, as we shall see, even the compressibility of gases may often be neglected.

Now the mass of an element of fluid is

\[
\delta m = \rho \, \delta V, \tag{8-124}
\]

and this will remain constant even though the volume and density may not:

\[
\frac{d}{dt} \delta m = \frac{d}{dt} (\rho \, \delta V) = 0. \tag{8-125}
\]

Let us carry out the differentiation, making use of Eq. (8-116):

\[
\delta V \frac{dp}{dt} + \rho \, \frac{d}{dt} \delta V = \delta V \frac{dp}{dt} + \rho \nabla \cdot \mathbf{v} \, \delta V = 0,
\]

or, when \( \delta V \) is divided out,

\[
\frac{dp}{dt} + \rho \nabla \cdot \mathbf{v} = 0. \tag{8-126}
\]

By utilizing Eq. (8-113), we can rewrite this in terms of the partial derivatives referred to a fixed point in space:

\[
\frac{dp}{dt} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = 0.
\]
The last two terms can be combined, using the properties of \( \nabla \) as a symbol of differentiation:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{8–127}
\]

This is the equation of continuity for the motion of continuous matter. It states essentially that matter is nowhere created or destroyed; the mass \( \delta m \) in any volume \( \delta V \) moving with the fluid remains constant.

We shall make frequent use in the remainder of this chapter of the properties of the symbol \( \nabla \), which were described briefly in Section 3–6. The operator \( \nabla \) has the algebraic properties of a vector and, in addition, when a product is involved, it behaves like a differentiation symbol. The simplest way to perform this sort of manipulation, when \( \nabla \) operates on a product, is first to write a sum of products in each of which only one factor is to be differentiated. The factor to be differentiated may be indicated by underlining it. Then each term may be manipulated according to the rules of vector algebra, except that the underlined factor must be kept behind the \( \nabla \) symbol. When the underlined factor is the only one behind the \( \nabla \) symbol, or when all other factors are separated out by parentheses, the underline may be omitted, as there is no ambiguity as to what factor is to be differentiated by the components of \( \nabla \). As an example, the relation between Eqs. (8–126) and (8–127) is made clear by the following computation:

\[
\nabla \cdot (\rho \mathbf{v}) = \nabla \cdot (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}) \\
= (\nabla \rho) \cdot \mathbf{v} + \rho \nabla \cdot \mathbf{v} \\
= (\nabla \rho) \cdot \mathbf{v} + \rho \nabla \cdot \mathbf{v} \\
= \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v}. \tag{8–128}
\]

Any formulas arrived at in this way can always be verified by writing out both sides in terms of components, and the reader should do this a few times to convince himself. However, it is usually far less work to make use of the properties of the \( \nabla \) symbol.

We now wish to calculate the rate of flow of mass through a surface \( S \) fixed in space. Let \( dS \) be an element of surface, and let \( \mathbf{n} \) be a unit vector normal to \( dS \). If we construct a cylinder by moving \( dS \) through a distance \( v \, dt \) in the direction of \( -\mathbf{v} \), then in a time \( dt \) all the matter in this cylinder will pass through the surface \( dS \) (Fig. 8–6). The amount of mass in this cylinder is

\[
\rho \mathbf{n} \cdot \mathbf{v} \, dt \, dS,
\]

where \( \mathbf{n} \cdot \mathbf{v} \, dt \) is the altitude perpendicular to the face \( dS \). The rate of flow of mass through a surface \( S \) is therefore

\[
\frac{dm}{dt} = \oint_S \rho \mathbf{n} \cdot \mathbf{v} \, dS = \int_S \mathbf{n} \cdot (\rho \mathbf{v}) \, dS. \tag{8–129}
\]
Fig. 8-6. Flow of fluid through a surface element.

If $\mathbf{n} \cdot \mathbf{v}$ is positive, the mass flow across $S$ is in the direction of $\mathbf{n}$; if $\mathbf{n} \cdot \mathbf{v}$ is negative, the mass flow is in the reverse direction. We see that $\rho \mathbf{v}$, the momentum density, is also the mass current, in the sense that its component in any direction gives the rate of mass flow per unit area in that direction. We can now give a further interpretation of Eq. (8-127) by integrating it over a fixed volume $V$ bounded by a surface $S$ with outward normal $\mathbf{n}$:

$$
\int \int \int \frac{\partial \rho}{\partial t} dV + \int \int \int \nabla \cdot (\rho \mathbf{v}) dV = 0. \tag{8-130}
$$

Since the volume $V$ here is a fixed volume, we can take the time differentiation outside the integral in the first term. If we apply Gauss’ divergence theorem to the second integral, we can rewrite this equation:

$$
\frac{d}{dt} \int \int \rho dV = - \int \int \mathbf{n} \cdot (\rho \mathbf{v}) dS. \tag{8-131}
$$

This equation states that the rate of increase of mass inside the fixed volume $V$ is equal to the negative of the rate of flow of mass outward across the surface. This result emphasizes the physical interpretation of each term in Eq. (8-127). In particular, the second term evidently represents the rate of flow of mass away from any point. Conversely, by starting with the self-evident equation (8-131) and working backwards, we have an independent derivation of Eq. (8-127).

Equations analogous to Eqs. (8-126), (8-127), (8-129), and (8-131) apply to the density, velocity, and rate of flow of any physical quantity. An equation of the form (8-127) applies, for example, to the flow of electric charge, if $\rho$ is the charge density and $\rho \mathbf{v}$ the electric current density.
The curl of the velocity \( \mathbf{v} \times \mathbf{v} \) is a concept which is useful in describing fluid flow. To understand its meaning, we compute the integral of the normal component of curl \( \mathbf{v} \) across a surface \( S \) bounded by a curve \( C \). By Stokes' theorem (3–117), this is

\[
\int_S \mathbf{n} \cdot (\mathbf{v} \times \mathbf{v}) \, dS = \int_C \mathbf{v} \cdot d\mathbf{r}, \tag{8–132}
\]

where the line integral is taken around \( C \) in the positive sense relative to the normal \( \mathbf{n} \), as previously defined. If the curve \( C \) surrounds a vortex in the fluid, so that \( \mathbf{v} \) is parallel to \( d\mathbf{r} \) around \( C \) (Fig. 8–7), then the line integral on the right is positive and measures, in a sense, the rate at which the fluid is whirling around the vortex. Thus \( \mathbf{v} \times \mathbf{v} \) is a sort of measure of the rate of rotation of the fluid per unit area; hence the name curl \( \mathbf{v} \). Curl \( \mathbf{v} \) has a nonzero value in the neighborhood of a vortex in the fluid. Curl \( \mathbf{v} \) may also be nonzero, however, in regions where there is no vortex, that is, where the fluid does not actually circle a point, provided there is a transverse velocity gradient. Figure 8–7 illustrates the two cases. In each case, the line integral of \( \mathbf{v} \) counterclockwise around the circle \( C \) will have a positive value. If the curl of \( \mathbf{v} \) is zero everywhere in a moving fluid, the flow is said to be irrotational. Irrotational flow is important chiefly because it presents fairly simple mathematical problems. If at any point \( \mathbf{v} \times \mathbf{v} = 0 \), then an element of fluid at that point will have no net angular velocity about that point, although its shape and size may be changing.

We arrive at a more precise meaning of curl \( \mathbf{v} \) by introducing a coordinate system rotating with angular velocity \( \omega \). If \( \mathbf{v}' \) designates the velocity of the fluid relative to the rotating system, then by Eq. (7–33),

\[
\mathbf{v} = \mathbf{v}' + \omega \times \mathbf{r},
\]

where \( \mathbf{r} \) is a vector from the axis of rotation (whose location does not matter in this discussion) to a point in the fluid. Curl \( \mathbf{v} \) is now
\[ \nabla \times \mathbf{v} = \nabla \times \mathbf{v}' + \nabla \times (\mathbf{\omega} \times \mathbf{r}) \]
\[ = \nabla \times \mathbf{v}' + \mathbf{\omega} \nabla \cdot \mathbf{r} - \mathbf{\omega} \cdot \nabla \mathbf{r} \]
\[ = \nabla \times \mathbf{v}' + 3\mathbf{\omega} - \mathbf{\omega} \]
\[ = \nabla \times \mathbf{v}' + 2\mathbf{\omega}, \]

where the second line follows from Eq. (3-35) for the triple cross product, and the third line by direct calculation of the components in the second and third terms. If we set
\[ \mathbf{\omega} = \frac{1}{2} \nabla \times \mathbf{v}, \]
then
\[ \nabla \times \mathbf{v}' = 0. \]

Thus if \( \nabla \times \mathbf{v} \neq 0 \) at a point \( P \), then in a coordinate system rotating with angular velocity \( \mathbf{\omega} = \frac{1}{2} \nabla \times \mathbf{v} \), the fluid flow is irrotational at the point \( P \). We may therefore interpret \( \frac{1}{2} \nabla \times \mathbf{v} \) as the angular velocity of the fluid near any point. If \( \nabla \times \mathbf{v} \) is constant, then it is possible to introduce a rotating coordinate system in which the flow is irrotational everywhere.

8–7 Equations of motion for an ideal fluid. For the remainder of this chapter, except in the last section, we shall consider the motion of an ideal fluid, that is, one in which there are no shearing stresses, even when the fluid is in motion. The stress within an ideal fluid consists in a pressure \( p \) alone. This is a much greater restriction in the case of moving fluids than in the case of fluids in equilibrium (Section 5–11). A fluid, by definition, supports no shearing stress when in equilibrium, but all fluids have some viscosity and therefore there are always some shearing stresses between layers of fluid in relative motion. An ideal fluid would have no viscosity, and our results for ideal fluids will therefore apply only when the viscosity is negligible.

Let us suppose that, in addition to the pressure, the fluid is acted on by a body force of density \( f \) per unit volume, so that the body force acting on a volume element \( \delta V \) of fluid is \( f \delta V \). We need, then, to calculate the force density due to pressure. Let us consider a volume element \( \delta V = \delta x \delta y \delta z \) in the form of a rectangular box (Fig. 8–8). The force due to pressure on the left face of the box is \( p \delta y \delta z \), and acts in the \( x \)-direction. The force due to pressure on the right face of the box is also \( p \delta y \delta z \), and acts in the opposite direction. Hence the net \( x \)-component of force \( \delta F_x \) on the box depends upon the difference in pressure between the left and right faces of the box:
\[ \delta F_x = \left( -\frac{\partial p}{\partial x} \delta x \right) \delta y \delta z. \]
A similar expression may be derived for the components of force in the y- and z-directions. The total force on the fluid in the box due to pressure is then

\[ \delta F = \left( -i \frac{\partial p}{\partial x} - j \frac{\partial p}{\partial y} - k \frac{\partial p}{\partial z} \right) \delta V \]

\[ = -\nabla p \delta V. \quad (8-136) \]

The force density per unit volume due to pressure is therefore \(-\nabla p\). This result was also obtained in Section 5-11 [Eq. (5-172)].

We can now write the equation of motion for a volume element \(\delta V\) of fluid:

\[ \rho \delta V \frac{d\mathbf{v}}{dt} = \mathbf{f} \delta V - \nabla p \delta V. \quad (8-137) \]

This equation is usually written in the form

\[ \rho \frac{d\mathbf{v}}{dt} + \nabla p = \mathbf{f}. \quad (8-138) \]

By making use of the relation (8-113), we may rewrite this in terms of derivatives at a fixed point:

\[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{1}{\rho} \nabla p = \frac{\mathbf{f}}{\rho}, \quad (8-139) \]

where \(\mathbf{f}/\rho\) is the body force per unit mass. This is Euler's equation of motion for a moving fluid.

If the density \(\rho\) depends only on the pressure \(p\), we shall call the fluid homogeneous. This definition does not imply that the density is uniform. An incompressible fluid is homogeneous if its density is uniform. A com-
pressible fluid of uniform chemical composition and uniform temperature throughout is homogeneous. When a fluid expands or contracts under the influence of pressure changes, work is done by or on the fluid, and part of this work may appear in the form of heat. If the changes in density occur sufficiently slowly so that there is adequate time for heat flow to maintain the temperature uniform throughout the fluid, the fluid may be considered homogeneous within the meaning of our definition. The relation between density and pressure is then determined by the equation of state of the fluid or by its isothermal bulk modulus (Section 5–11). In some cases, changes in density occur so rapidly that there is no time for any appreciable flow of heat. In such cases the fluid may also be considered homogeneous, and the adiabatic relation between density and pressure or the adiabatic bulk modulus should be used. In cases between these two extremes, the density will depend not only on pressure, but also on temperature, which, in turn, depends upon the rate of heat flow between parts of the fluid at different temperatures.

In a homogeneous fluid, there are four unknown functions to be determined at each point in space and time, the three components of velocity \( v \), and the pressure \( p \). We have, correspondingly, four differential equations to solve, the three components of the vector equation of motion (8–139), and the equation of continuity (8–127). The only other quantities appearing in these equations are the body force, which is assumed to be given, and the density \( \rho \), which can be expressed as a function of the pressure. Of course, Eqs. (8–139) and (8–127) have a tremendous variety of solutions. In a specific problem we would need to know the conditions at the boundary of the region in which the fluid is moving and the values of the functions \( v \) and \( p \) at some initial instant. In the following sections, we shall confine our attention to homogeneous fluids. In the intermediate case mentioned at the end of the last paragraph, where the fluid is inhomogeneous and the density depends on both pressure and temperature, we have an additional unknown function, the temperature, and we will need an additional equation determined by the law of heat flow. We shall not consider this case, although it is a very important one in many problems.

8–8 Conservation laws for fluid motion. Inasmuch as the laws of fluid motion are derived from Newton’s laws of motion, we may expect that appropriate generalizations of the conservation laws of momentum, energy, and angular momentum also hold for fluid motion. We have already had an example of a conservation law for fluid motion, namely, the equation of continuity [Eq. (8–127) or (8–131)], which expresses the law of conservation of mass. Mass is conserved also in particle mechanics, but we did not find it necessary to write an equation expressing this fact.
A conservation law in fluid mechanics may be written in many equivalent forms. It will be instructive to study some of these in order to get a clearer idea of the physical meaning of the various mathematical expressions involved. Let \( \rho \) be the density of any physical quantity: mass, momentum, energy, or angular momentum. Then the simplest form of the conservation law for this quantity will be equation (8–125), which states that the amount of this quantity in an element \( \delta V \) of fluid remains constant. If the quantity in question is being produced at a rate \( Q \) per unit volume, then Eq. (8–125) should be generalized:

\[
\frac{d}{dt} (\rho \delta V) = Q \delta V. \tag{8–140}
\]

This is often called a conservation law for the quantity \( \rho \). It states that this quantity is appearing in the fluid at a rate \( Q \) per unit volume, or disappearing if \( Q \) is negative. In the sense in which we have used the term in Chapter 4, this should not be called a conservation law except when \( Q = 0 \). By a derivation exactly like that which led to Eq. (8–127), we can rewrite Eq. (8–140) as a partial differential equation:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = Q. \tag{8–141}
\]

This is probably the most useful form of conservation law. The meaning of the terms in Eq. (8–141) is brought out by integrating each term over a fixed volume \( V \) and using Gauss' theorem,\(^*\) as in the derivation of Eq. (8–131):

\[
\frac{d}{dt} \iiint_{V} \rho \, dV + \iint_{S} \mathbf{n} \cdot \rho \mathbf{v} \, dS = \iiint_{V} Q \, dV. \tag{8–142}
\]

According to the discussion preceding Eq. (8–129), this equation states that the rate of increase of the quantity within \( V \), plus the rate of flow outward across the boundary \( S \), equals the rate of appearance due to sources within \( V \). Another form of the conservation law which is sometimes useful is obtained by summing equation (8–140) over a volume \( V \) moving with the fluid:

\[
\sum \frac{d}{dt} (\rho \delta V) = \frac{d}{dt} \sum \rho \delta V = \sum Q \delta V. \tag{8–143}
\]

\(^*\) If \( \rho \) is a vector, as in the case of linear or angular momentum density, then a generalized form of Gauss' theorem [mentioned in Section 5–11 in connection with Eq. (5–178)] must be used.
If we pass to the limit \( \delta V \to 0 \), the summations become integrations:

\[
\frac{d}{dt} \iint_V \rho \, dV = \iint_V \int \int Q \, dV. \tag{8-144}
\]

The surface integral which appears in the left member of Eq. (8-142) does not appear in Eq. (8-144); since the volume \( V \) moves with the fluid, there is no flow across its boundary. Since Eqs. (8-140), (8-141), (8-142), and (8-144) are all equivalent, it is sufficient to derive a conservation law in any one of these forms. The others then follow. Usually it is easiest to derive an equation of the form (8-140), starting with the equation of motion in the form (8-138). We can also start with Eq. (8-139) and derive a conservation equation in the form (8-141), but a bit more manipulation is usually required.

In order to derive a conservation law for linear momentum, we first note that the momentum in a volume element \( \delta V \) is \( \rho v \delta V \). The momentum density per unit volume is therefore \( \rho v \), and this quantity will play the role played by \( \rho \) in the discussion of the preceding paragraph. In order to obtain an equation analogous to Eq. (8-140), we start with the equation of motion in the form (8-138), which refers to a point moving with the fluid, and multiply through by the volume \( \delta V \) of a small fluid element:

\[
\rho \delta V \frac{dv}{dt} + \nabla p \delta V = f \delta V. \tag{8-145}
\]

Since \( \rho \delta V = \delta m \) is constant, we may include it in the time derivative:

\[
\frac{d}{dt} (\rho v \delta V) = (f - \nabla p) \delta V. \tag{8-146}
\]

The momentum of a fluid element, unlike its mass, is not, in general, constant. This equation states that the time rate of change of momentum of a moving fluid element is equal to the body force plus the force due to pressure acting upon it. The quantity \( f - \nabla p \) here plays the role of \( Q \) in the preceding general discussion. Equation (8-146) can be rewritten in any of the forms (8-141), (8-142), and (8-144). For example, we may write it in the form (8-144):}

\[
\frac{d}{dt} \iint_V \rho v \, dV = \iint_V \int \int f \, dV - \iint_V \int \nabla p \, dV. \tag{8-147}
\]

We can now apply the generalized form of Gauss' theorem [Eq. (5-178)] to the second term on the right, to obtain
\[ \frac{d}{dt} \iiint_V \rho v \, dV = \iint_V \int \mathbf{f} \, dV + \int_S -np \, dS, \quad (8-148) \]

where \( S \) is the surface bounding \( V \).

This equation states that the time rate of change of the total linear momentum in a volume \( V \) of moving fluid is equal to the total external force acting on it. This result is an immediate generalization of the linear momentum theorem (4-7) for a system of particles. The internal forces, in the case of a fluid, are represented by the pressure within the fluid. By the application of Gauss' theorem, we have eliminated the pressure within the volume \( V \), leaving only the external pressure across the surface of \( V \). It may be asked how we have managed to eliminate the internal forces without making explicit use of Newton's third law, since Eq. (8-138), from which we started, is an expression only of Newton's first two laws. The answer is that the concept of pressure itself contains Newton's third law implicitly, since the force due to pressure exerted from left to right across any surface element is equal and opposite to the force exerted from right to left across the same surface element. Furthermore, the points of application of these two forces are the same, namely, at the surface element. Both forces necessarily have the same line of action, and there is no distinction between the weak and strong forms of Newton's third law. The internal pressures will therefore also be expected to cancel out in the equation for the time rate of change of angular momentum. A similar remark applies to the forces due to any kind of stresses in a fluid or a solid; Newton's third law in strong form is implicitly contained in the concept of stress.

Equations representing the conservation of angular momentum, analogous term by term with Eqs. (8-140) through (8-144), can be derived by taking the cross product of the vector \( \mathbf{r} \) with either Eq. (8-138) or (8-139), and suitably manipulating the terms. The vector \( \mathbf{r} \) is here the vector from the origin about which moments are to be computed to any point in the moving fluid or in space. This development is left as an exercise. The law of conservation of angular momentum is responsible for the vortices formed when a liquid flows out through a small hole in the bottom of a tank. The only body force here is gravity, which exerts no torque about the hole, and it can be shown that if the pressure is constant, or depends only on vertical depth, there is no net vertical component of torque across any closed surface due to pressure. Therefore the angular momentum of any part of the fluid remains constant. If a fluid element has any angular momentum at all initially, when it is some distance from the hole, its angular velocity will have to increase in inverse proportion to the square of its distance from the hole in order for its angular momentum to remain constant as it approaches the hole.
In order to derive a conservation equation for the energy, we take the dot product of \( \mathbf{v} \) with Eq. (8-146), to obtain

\[
\frac{d}{dt} \left( \frac{1}{2} \rho \mathbf{v}^2 \right) \mathbf{\delta V} = \mathbf{v} \cdot (\mathbf{f} - \nabla \rho) \mathbf{\delta V}. \tag{8-149}
\]

This is the energy theorem in the form (8-140). In place of the density \( \rho \), we have here the kinetic energy density \( \frac{1}{2} \rho \mathbf{v}^2 \). The rate of production of kinetic energy per unit volume is

\[
\mathbf{Q} = \mathbf{v} \cdot (\mathbf{f} - \nabla \rho). \tag{8-150}
\]

In analogy with our procedure in particle mechanics, we shall now try to define additional forms of energy so as to include as much as possible of the right member of Eq. (8-149) under the time derivative on the left. We can see how to rewrite the second term on the right by making use of Eqs. (8-113) and (8-116):

\[
\frac{d}{dt} (p \mathbf{\delta V}) = \frac{dp}{dt} \mathbf{\delta V} + p \frac{d}{dt} \mathbf{\delta V}
\]

\[
= \frac{\partial p}{\partial t} \mathbf{\delta V} + \mathbf{v} \cdot \nabla p \mathbf{\delta V} + p \nabla \cdot \mathbf{v} \mathbf{\delta V}, \quad \tag{8-151}
\]

so that

\[
-\mathbf{v} \cdot \nabla p \mathbf{\delta V} = -\frac{d}{dt} (p \mathbf{\delta V}) + \frac{\partial p}{\partial t} \mathbf{\delta V} + p \nabla \cdot \mathbf{v} \mathbf{\delta V}. \quad \tag{8-152}
\]

Let us now assume that the body force \( \mathbf{f} \) is a gravitational force:

\[
\mathbf{f} = \rho g = \rho \nabla \mathcal{G}, \quad \tag{8-153}
\]

where \( \mathcal{G} \) is the gravitational potential [Eq. (6-16)], i.e., the negative potential energy per unit mass due to gravitation. The first term on the right in Eq. (8-149) is then

\[
\mathbf{v} \cdot \mathbf{f} \mathbf{\delta V} = (\mathbf{v} \cdot \nabla \mathcal{G}) \rho \mathbf{\delta V} = \left( \frac{d\mathcal{G}}{dt} - \frac{\partial \mathcal{G}}{\partial t} \right) \rho \mathbf{\delta V}
\]

\[
= \frac{d}{dt} (\rho \mathcal{G} \mathbf{\delta V}) - \rho \frac{\partial \mathcal{G}}{\partial t} \mathbf{\delta V}, \quad \tag{8-154}
\]

since \( \rho \mathbf{\delta V} = \rho \mathbf{\delta m} \) is constant. With the help of Eqs. (8-152) and (8-154), Eq. (8-149) can be rewritten:

\[
\frac{d}{dt} \left[ \frac{1}{2} \rho \mathbf{v}^2 + p - \rho \mathcal{G} \right] \mathbf{\delta V} = \left( \frac{\partial p}{\partial t} - \rho \frac{\partial \mathcal{G}}{\partial t} \right) \mathbf{\delta V} + p \nabla \cdot \mathbf{v} \mathbf{\delta V}. \quad \tag{8-155}
\]

The pressure \( p \) here plays the role of a potential energy density whose
negative gradient gives the force density due to pressure [Eq. (8–136)]. The time rate of change of kinetic energy plus gravitational potential energy plus potential energy due to pressure is equal to the expression on the right.

Ordinarily, the gravitational field at a fixed point in space will not change with time (except perhaps in applications to motions of gas clouds in astronomical problems). If the pressure at a given point in space is constant also; then the first term on the right vanishes. What is the significance of the second term? For an incompressible fluid, \( \nabla \cdot \mathbf{v} = 0 \), and the second term would vanish also. We therefore suspect that it represents energy associated with compression and expansion of the fluid element \( \delta V \). Let us check this hypothesis by calculating the work done in changing the volume of the element \( \delta V \). The work \( dW \) done by the fluid element \( \delta V \), through the pressure which it exerts on the surrounding fluid when it expands by an amount \( d \delta V \), is

\[
dW = pd \delta V. \tag{8–156}
\]

The rate at which energy is supplied by the expansion of the fluid element is, by Eq. (8–116),

\[
\frac{dW}{dt} = p \frac{d \delta V}{dt} = p \nabla \cdot \mathbf{v} \delta V, \tag{8–157}
\]

which is just the last term in Eq. (8–155). So far, all our conservation equations are valid for any problem involving ideal fluids. If we restrict ourselves to homogeneous fluids, that is, fluids whose density depends only on the pressure, we can define a potential energy associated with the expansion and contraction of the fluid element \( \delta V \). We shall define the potential energy \( u \delta m \) on the fluid element \( \delta V \) as the negative work done through its pressure on the surrounding fluid when the pressure changes from a standard pressure \( p_0 \) to any pressure \( p \). The potential energy per unit mass \( u \) will then be a function of \( p \):

\[
u \delta m = - \int_{p_0}^{p} pd \delta V. \tag{8–158}
\]

The volume \( \delta V = \delta m/\rho \) is a function of pressure, and we may rewrite this in various forms:

\[
u = \int_{p_0}^{p} \frac{p \, dp}{\rho^2} = \int_{p_0}^{p} \frac{\rho \, dp}{\rho^2} \, dp \tag{8–159}
\]

\[
= \int_{p_0}^{p} \frac{p}{\rho B} \, dp,
\]
where the last step makes use of the definition of the bulk modulus [Eq. (5–116)]. The time rate of change of \( u \) is, by Eqs. (8–158) or (8–159) and (8–116),

\[
\frac{d(u \, \delta m)}{dt} = -p \, \frac{d \delta V}{dt} = -p \mathbf{\nabla} \cdot \mathbf{v} \, \delta V.
\] (8–160)

We can now include the last term on the right in Eq. (8–155) under the time derivative on the left:

\[
\frac{d}{dt} \left[ \left( \frac{1}{2} \rho v^2 + p - \rho g + \rho u \right) \delta V \right] = \left( \frac{\partial p}{\partial t} - \rho \frac{\partial g}{\partial t} \right) \delta V.
\] (8–161)

The interpretation of this equation is clear from the preceding discussion. It can be rewritten in any of the forms (8–141), (8–142), and (8–144).

If \( p \) and \( g \) are constant at any fixed point in space, then the total kinetic plus potential energy of a fluid element remains constant as it moves along. It is convenient to divide by \( \delta m = \rho \, \delta V \) in order to eliminate reference to the volume element:

\[
\frac{d}{dt} \left( \frac{v^2}{2} + \frac{p}{\rho} - g + u \right) = \frac{1}{\rho} \frac{\partial p}{\partial t} - \frac{\partial g}{\partial t}.
\] (8–162)

This is Bernoulli's theorem. The term \( \partial g/\partial t \) is practically always zero; we have kept it merely to make clear the meaning of the term \((1/\rho)(\partial \rho/\partial t)\), which plays a similar role and is not always zero. When both terms on the right are zero, as in the case of steady flow, we have, for a point moving along with the fluid,

\[
\frac{v^2}{2} + \frac{p}{\rho} - g + u = \text{a constant.}
\] (8–163)

Other things being equal, that is if \( u, g, \) and \( \rho \) are constant, the pressure of a moving fluid decreases as the velocity increases. For an incompressible fluid, \( \rho \) and \( u \) are necessarily constant.

The conservation laws of linear and angular momentum apply not only to ideal fluids, but also, when suitably formulated, to viscous fluids and even to solids, in view of the remarks made above regarding Newton's third law and the concept of stress. The law of conservation of energy (8–162) will not apply, however, to viscous fluids, since the viscosity is due to an internal friction which results in a loss of kinetic and potential energies, unless conversion of mechanical to heat energy by viscous friction is included in the law. [Equation (8–155) applies in any case.]

8-9 Steady flow. By steady flow of a fluid we mean a motion of the fluid in which all quantities associated with the fluid, velocity, density, pressure, force density, etc., are constant in time at any given point in
space. For steady flow, all partial derivatives with respect to time can be set equal to zero. The total time derivative, which designates the time rate of change of a quantity relative to a point moving with the fluid, will not in general be zero, but, by Eq. (8–113), will be

$$\frac{d}{dt} = \mathbf{v} \cdot \nabla.$$  \hspace{1cm} (8–164)

The path traced out by any fluid element as it moves along is called a streamline. A streamline is a line which is parallel at each point \((x, y, z)\) to the velocity \(\mathbf{v}(x, y, z)\) at that point. The entire space within which the fluid is flowing can be filled with streamlines such that through each point there passes one and only one streamline. If we introduce along any streamline a coordinate \(s\) which represents the distance measured along the streamline from any fixed point, we can regard any quantity associated with the fluid as a function of \(s\) along the streamline. The component of the symbol \(\nabla\) along the streamline at any point is \(d/ds\), as we see if we choose a coordinate system whose \(x\)-axis is directed along the streamline at that point. Equation (8–164) can therefore be rewritten:

$$\frac{d}{dt} = \mathbf{v} \frac{d}{ds}. \hspace{1cm} (8–165)$$

This equation is also evident from the fact that \(v = ds/dt\). For example, Eq. (8–162), in the case of steady flow, can be written:

$$\frac{d}{ds} \left( \frac{v^2}{2} + \frac{p}{\rho} - g + u \right) = 0. \hspace{1cm} (8–166)$$

The quantity in parentheses is therefore constant along a streamline.

The equation of continuity (8–127) in the case of steady flow becomes

$$\nabla \cdot (\rho \mathbf{v}) = 0. \hspace{1cm} (8–167)$$

If we integrate this equation over a fixed volume \(V\), and apply Gauss' theorem, we have

$$\int_S \mathbf{n} \cdot (\rho \mathbf{v}) dS = 0, \hspace{1cm} (8–168)$$

where \(S\) is the closed surface bounding \(V\). This equation simply states that the total mass flowing out of any closed surface is zero.

If we consider all the streamlines which pass through any (open) surface \(S\), these streamlines form a tube, called a tube of flow (Fig. 8–9). The walls of a tube of flow are everywhere parallel to the streamlines, so that no fluid enters or leaves it. A surface \(S\) which is drawn everywhere perpendicular to the streamlines and through which passes each streamline in
a tube of flow, will be called a cross section of the tube. If we apply Eq. (8–168) to the closed surface bounded by the walls of a tube of flow and two cross sections \( S_1 \) and \( S_2 \), then since \( n \) is perpendicular to \( v \) over the walls of the tube, and \( n \) is parallel or antiparallel to \( v \) over the cross sections, we have

\[
\int \int_{S_1} \rho v \, dS - \int \int_{S_2} \rho v \, dS = 0, \tag{8–169}
\]

or

\[
\int \int_S \rho v \, dS = I = \text{a constant}, \tag{8–170}
\]

where \( S \) is any cross section along a given tube of flow. The constant \( I \) is called the fluid current through the tube.

The energy conservation equation (8–161), when rewritten in the form (8–141), becomes, in the case of steady flow,

\[
\nabla \cdot [(\frac{1}{2} \rho v^2 + p - \rho g + \rho u) v] = 0. \tag{8–171}
\]

This equation has the same form as Eq. (8–167), and we can conclude in the same way that the energy current is the same through any cross section \( S \) of a tube of flow:

\[
\int \int_S (\frac{1}{2} \rho v^2 + p - \rho g + \rho u) v \, dS = \text{a constant}. \tag{8–172}
\]

This result is closely related to Eq. (8–166).

If the flow is not only steady, but also irrotational, then

\[
\nabla \times v = 0 \tag{8–173}
\]

everywhere. This equation is analogous in form to Eq. (3–189) for a conservative force, and we can proceed as in Section 3–12 to show that if Eq. (8–173) holds, it
is possible to define a velocity potential function \( \phi(x, y, z) \) by the equation

\[
\phi(r) = \int_{r_x}^{r} \mathbf{v} \cdot dr,
\]

(8–174)

where \( r_x \) is any fixed point. The velocity at any point will then be

\[
\mathbf{v} = \nabla \phi.
\]

(8–175)

Substituting this in Eq. (8–167), we have an equation to be solved for \( \phi \):

\[
\nabla \cdot (\rho \nabla \phi) = 0.
\]

(8–176)

In the cases usually studied, the fluid can be considered incompressible, and this becomes

\[
\nabla^2 \phi = 0.
\]

(8–177)

This equation is identical in form with Laplace's equation (6–35) for the gravitational potential in empty space. Hence the techniques of potential theory may be used to solve problems involving irrotational flow of an incompressible fluid.

**8–10 Sound waves.** Let us assume a fluid at rest with pressure \( p_0 \), density \( \rho_0 \), in equilibrium under the action of a body force \( f_0 \), constant in time. Equation (8–139) then becomes

\[
\frac{1}{\rho_0} \nabla p_0 = \frac{f_0}{\rho_0}.
\]

(8–178)

We may note that this equation agrees with Eq. (5–172) deduced in Section 5–11 for a fluid in equilibrium. Let us now suppose that the fluid is subject to a small disturbance, so that the pressure and density at any point become

\[
p = p_0 + p',
\]

(8–179)

\[
\rho = \rho_0 + \rho',
\]

(8–180)

where \( p' \ll p \) and \( \rho' \ll \rho \). We assume that the resulting velocity \( \mathbf{v} \) and its space and time derivatives are everywhere very small. If we substitute Eqs. (8–179) and (8–180) in the equation of motion (8–139), and neglect higher powers than the first of \( p', \rho', \mathbf{v} \) and their derivatives, making use of Eq. (8–178), we obtain

\[
\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_0} \nabla p'.
\]

(8–181)

Making a similar substitution in Eq. (8–127), we obtain

\[
\frac{\partial \rho'}{\partial t} = -\rho_0 \nabla \cdot \mathbf{v} - \mathbf{v} \cdot \nabla \rho_0.
\]

(8–182)
Let us assume that the equilibrium density \( \rho_0 \) is uniform, or nearly so, so that \( \nabla \rho_0 \) is zero or very small, and the second term can be neglected.

The pressure increment \( p' \) and density increment \( \rho' \) are related by the bulk modulus according to Eq. (5–183):

\[
\frac{\rho'}{\rho_0} = \frac{p'}{B}.
\]  

(8–183)

This equation may be used to eliminate either \( \rho' \) or \( p' \) from Eqs. (8–181) and (8–182). Let us eliminate \( \rho' \) from Eq. (8–182):

\[
\frac{\partial p'}{\partial t} = -B \nabla \cdot \mathbf{v}.
\]

(8–184)

Equations (8–181) and (8–184) are the fundamental differential equations for sound waves. The analogy with the form (8–101) for one-dimensional waves is apparent. Here again we have two quantities, \( p' \) and \( \mathbf{v} \), such that the time derivative of either is proportional to the space derivatives of the other. In fact, if \( \mathbf{v} = \mathbf{i}v_x \) and if \( v_x \) and \( p' \) are functions of \( x \) alone, then Eqs. (8–181) and (8–184) reduce to Eqs. (8–101).

We may proceed, in analogy with the discussion in Section 8–5, to eliminate either \( \mathbf{v} \) or \( p' \) from these equations. In order to eliminate \( \mathbf{v} \), we take the divergence of Eq. (8–181) and interchange the order of differentiation, again assuming \( \rho_0 \) nearly uniform:

\[
\frac{\partial}{\partial t} (\nabla \cdot \mathbf{v}) = -\frac{1}{\rho_0} \nabla^2 p'.
\]

(8–185)

We now differentiate Eq. (8–184) with respect to \( t \), and substitute from Eq. (8–185):

\[
\nabla^2 p' - \frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2} = 0,
\]

(8–186)

where

\[
c = \left( \frac{B}{\rho_0} \right)^{1/2}.
\]

(8–187)

This is the three-dimensional wave equation, as we shall show presently. Formula (8–187) for the speed of sound waves was first derived by Isaac Newton, and applies either to liquids or gases. For gases, Newton assumed that the isothermal bulk modulus \( B = \rho \) should be used, but Eq. (8–187) does not then agree with the experimental values for the speed of sound. The sound vibrations are so rapid that they should be treated as adiabatic, and the adiabatic bulk modulus \( B = \gamma p \) should be used, where \( \gamma \) is the ratio of specific heat at constant pressure to that at constant
volume.* Formula (8–187) then agrees with the experimental values of \( c \).

If we eliminate \( p' \) by a similar process, we obtain a wave equation for \( \mathbf{v} \):

\[
\nabla^2 \mathbf{v} - \frac{1}{c^2} \frac{\partial^2 \mathbf{v}}{\partial t^2} = 0.
\]

(8–188)

In deriving Eq. (8–188), it is necessary to use the fact that \( \nabla \times \mathbf{v} = 0 \). It follows from Eq. (8–181) that \( \nabla \mathbf{v} \) is in any case independent of time, so that the time-dependent part of \( \mathbf{v} \) which is present in a sound wave is irrotational. [We could add to the sound wave a small steady flow with \( \nabla \times \mathbf{v} \neq 0 \), without violating Eqs. (8–181) and (8–182).]

In order to show that Eq. (8–186) leads to sound waves traveling with speed \( c \), we note first that if \( p' \) is a function of \( x \) and \( t \) alone, Eq. (8–186) becomes

\[
\frac{\partial^2 p'}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2} = 0.
\]

(8–189)

This is of the same form as the one-dimensional wave equation (8–6), and therefore has solutions of the form

\[
p' = f(x - ct).
\]

(8–190)

This is called a plane wave, for at any time \( t \) the phase \( x - ct \) and the pressure \( p' \) are constant along any plane \( (x = \) a constant) parallel to the \( yz \)-plane. A plane wave traveling in the direction of the unit vector \( \mathbf{n} \) will be given by

\[
p' = f(\mathbf{n} \cdot \mathbf{r} - ct),
\]

(8–191)

where \( \mathbf{r} \) is the position vector from the origin to any point in space. To see that this is a wave in the direction \( \mathbf{n} \), we rotate the coordinate system until the \( x \)-axis lies in this direction, in which case Eq. (8–191) reduces to Eq. (8–190). The planes \( f = \) a constant, at any time \( t \), are now perpendicular to \( \mathbf{n} \), and travel in the direction of \( \mathbf{n} \) with velocity \( c \). We can see from the argument just given that the solution (8–191) must satisfy Eq. (8–186), or we may verify this by direct computation, for any coordinate system:

\[
\nabla p' = \frac{df}{d\xi} \nabla \xi = \frac{df}{d\xi} \mathbf{n},
\]

(8–192)

where

\[
\xi = \mathbf{n} \cdot \mathbf{r} - ct,
\]

(8–193)

---

and, similarly,
\[
\nabla^2 p' = \frac{d^2 f}{d\xi^2} n \cdot \nabla \xi = \frac{d^2 f}{d\xi^2} n \cdot n = \frac{d^2 f}{d\xi^2},
\]
(8–194)

\[
\frac{\partial^2 p'}{\partial t^2} = \frac{d^2 f}{d\xi^2} \left(\frac{\partial \xi}{\partial t}\right)^2 = c^2 \frac{d^2 f}{d\xi^2},
\]
(8–195)

so that Eq. (8–186) is satisfied, no matter what the function \( f(\xi) \) may be.

Equation (8–188) will also have plane wave solutions:
\[
v = h(n' \cdot r - ct),
\]
(8–196)

corresponding to waves traveling in the direction \( n' \) with velocity \( c \), where \( h \) is a vector function of \( \xi' = n' \cdot r - ct \). To any given pressure wave of the form (8–191) will correspond a velocity wave of the form (8–196), related to it by Eqs. (8–181) and (8–182). If we calculate \( \partial v / \partial t \) from Eq. (8–196), and \( \nabla p' \) from Eq. (8–191), and substitute in Eq. (8–181), we will have
\[
\frac{dh}{d\xi'} = \frac{n}{(B \rho_0)^{1/2}} \frac{df}{d\xi}.
\]
(8–197)

Equation (8–197) must hold at all points \( r \) at all times \( t \). The right member of this equation is a function of \( \xi \) and is constant for a constant \( \xi \). Consequently, the left member must be constant when \( \xi \) is constant, and must be a function only of \( \xi \), which implies that \( \xi' = \xi \) (or at least that \( \xi' \) is a function of \( \xi \)), and hence \( n' = n \). This is obvious physically, that the velocity wave must travel in the same direction as the pressure wave.

We can now set \( \xi' = \xi \), and solve Eq. (8–197) for \( h \):
\[
h = \frac{n}{(B \rho_0)^{1/2}} f,
\]
(8–198)

where the additive constant is zero, since both \( p' \) and \( v \) are zero in a region where there is no disturbance. Equations (8–198), (8–196), and (8–190) imply that for a plane sound wave traveling in the direction \( n \), the pressure increment and velocity are related by the equation
\[
v = \frac{p'}{(B \rho_0)^{1/2}} n,
\]
(8–199)

where \( v \), of course, is here the velocity of a fluid particle, not that of the wave, which is \( cn \). The velocity of the fluid particles is along the direction of propagation of the sound wave, so that sound waves in a fluid are longitudinal. This is a consequence of the fact that the fluid will not support a shearing stress, and is not true of sound waves in a solid, which may be either longitudinal or transverse.
A plane wave oscillating harmonically in time with angular frequency $\omega$ may be written in the form

$$p' = A \cos (k \cdot r - \omega t) = \text{Re} \, Ae^{i(k \cdot r - \omega t)}, \quad (8-200)$$

where $k$, the wave vector, is given by

$$k = \frac{\omega}{c} n. \quad (8-201)$$

If we consider a surface perpendicular to $n$ which moves back and forth with the fluid as the wave goes by, the work done by the pressure across this surface in the direction of the pressure is, per unit area per unit time,

$$P = p v. \quad (8-202)$$

If $v$ oscillates with average value zero, then since $p = p_0 + p'$, where $p_0$ is constant, the average power is

$$P_{av} = \langle p'v \rangle_{av} = \frac{\langle (p')^2 \rangle_{av}}{\rho_0 B^{1/2}}, \quad (8-203)$$

where we have made use of Eq. (8-199). This gives the amount of energy per unit area per second traveling in the direction $n$.

The three-dimensional wave equation (8-186) has many other solutions corresponding to waves of various forms whose wave fronts (surfaces of constant phase) are of various shapes, and traveling in various directions. As an example, we consider a spherical wave traveling out from the origin. The rate of energy flow is proportional to $p'r^2$ (a small portion of a spherical wave may be considered plane), and we expect that the energy flow per unit area must fall off inversely as the square of the distance, by the energy conservation law. Therefore $p'$ should be inversely proportional to the distance $r$ from the origin. We are hence led to try a wave of the form

$$p' = \frac{1}{r} f(r - ct). \quad (8-204)$$

This will represent a wave of arbitrary time-dependence, whose wave fronts, $\xi = r - ct = \text{a constant}$, are spheres expanding with the velocity $c$. It can readily be verified by direct computation, using either rectangular coordinates, or using spherical coordinates with the help of Eq. (3-124), that the solution (8-204) satisfies the wave equation (8-186).

A slight difficulty is encountered with the above development if we attempt to apply to a sound wave the expressions for energy flow and mass flow developed in the two preceding sections. The rate of flow of mass per unit area per second,
by Eqs. (8–199), (8–180), and (8–183), is

\[ \rho v = \rho_0 \left( 1 + \frac{p'}{B} \right) \frac{p'}{(\rho_0 B)^{1/2}} n. \]

We should expect that \( \rho v \) would be an oscillating quantity whose average value is zero for a sound wave, since there should be no net flow of fluid. If we average the above expression, we have

\[ \langle \rho v \rangle_{av} = \frac{\rho_0^{1/2}}{B^{3/2}} \left( \langle p'^2 \rangle_{av} + B \langle p' \rangle_{av} \right)n, \]

so that there is a small net flow of fluid in the direction of the wave, unless

\[ \langle p' \rangle_{av} = -\frac{\langle p'^2 \rangle_{av}}{B}. \tag{8–205} \]

If Eq. (8–205) holds, so that there is no net flow of fluid, then it can be shown that, to second-order terms in \( p' \) and \( v \), the energy current density given by Eq. (8–161) is, on the average, for a sound wave,

\[ \langle (\frac{1}{2} p v^2 + p - \rho_0 + \rho u) v \rangle_{av} = \frac{\langle p'^2 \rangle_{av}}{(\rho_0 B)^{1/2}} n, \tag{8–206} \]

in agreement with Eq. (8–203). When approximations are made in the equations of motion, we may expect that the solutions will satisfy the conservation laws only to the same degree of approximation. By adding second-order (or higher) terms like (8–205) to a first-order solution, we can of course satisfy the conservation laws to second-order terms (or higher).

8–11 Normal vibrations of fluid in a rectangular box. The problem of the vibrations of a fluid confined within a rigid box is of interest not only because of its applications to acoustical problems, but also because the methods used can be applied to problems in electromagnetic vibrations, vibrations of elastic solids, wave mechanics, and all phenomena in physics which are described by wave equations. In this section, we consider a fluid confined to a rectangular box of dimensions \( L_x L_y L_z \).

We proceed as in the solution of the one-dimensional wave equation in Section 8–2. We first assume a solution of Eq. (8–186) of the form

\[ p' = U(x, y, z) \Theta(t). \tag{8–207} \]

Substitution in Eq. (8–186) leads to the equation

\[ \frac{1}{U} \nabla^2 U = \frac{1}{c^2 \Theta} \frac{d^2 \Theta}{dt^2}. \tag{8–208} \]

Again we argue that since the left side depends only on \( x, y, \) and \( z \), and the
right side only on \( t \), both must be equal to a constant, which we shall call \(-\omega^2/c^2\):

\[
\frac{d^2\Theta}{dt^2} + \omega^2\Theta = 0, \quad \text{(8–209)}
\]

\[
\nabla^2 U + \frac{\omega^2}{c^2} U = 0. \quad \text{(8–210)}
\]

The solution of Eq. (8–209) can be written:

\[
\Theta = A \cos \omega t + B \sin \omega t, \quad \text{(8–211)}
\]

or

\[
\Theta = Ae^{-i\omega t}, \quad \text{(8–212)}
\]

where \( A \) and \( B \) are constant. The form (8–212) leads to traveling waves of the form (8–200). We are concerned here with standing waves, and we therefore choose the form (8–211). In order to solve Eq. (8–210), we again use the method of separation of variables, and assume that

\[
U(x, y, z) = X(x)Y(y)Z(z). \quad \text{(8–213)}
\]

Substitution in Eq. (8–210) leads to the equation

\[
\frac{1}{X} \frac{d^2X}{dx^2} + \frac{1}{Y} \frac{d^2Y}{dy^2} + \frac{1}{Z} \frac{d^2Z}{dz^2} = -\frac{\omega^2}{c^2}. \quad \text{(8–214)}
\]

This can hold for all \( x, y, z \) only if each term on the left is constant. We shall call these constants \(-k_x^2, -k_y^2, -k_z^2\), so that

\[
\frac{d^2X}{dx^2} + k_x^2 X = 0, \quad \frac{d^2Y}{dy^2} + k_y^2 Y = 0, \quad \frac{d^2Z}{dz^2} + k_z^2 Z = 0, \quad \text{(8–215)}
\]

where

\[
k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}. \quad \text{(8–216)}
\]

The solutions of Eqs. (8–215) in which we are interested are

\[
X = C_x \cos k_x x + D_x \sin k_x x,
\]

\[
Y = C_y \cos k_y y + D_y \sin k_y y, \quad \text{(8–217)}
\]

\[
Z = C_z \cos k_z z + D_z \sin k_z z.
\]

If we choose complex exponential solutions for \( X, Y, Z, \) and \( \Theta \), we arrive at the traveling wave solution (8–200), where \( k_x, k_y, k_z \) are the components of the wave vector \( k \).
We must now determine the appropriate boundary conditions to be applied at the walls of the box, which we shall take to be the six planes $x = 0, x = L_x, y = 0, y = L_y, z = 0, z = L_z$. The condition is evidently that the component of velocity perpendicular to the wall must vanish at the wall. At the wall $x = 0$, for example, $v_x$ must vanish. According to Eq. (8–181),

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x}. \quad (8–181)$$

We substitute for $p'$ from Eqs. (8–207), (8–211), (8–213), and (8–217):

$$\frac{\partial v_x}{\partial t} = -\frac{k_x Y Z}{\rho_0} (A \cos \omega t + B \sin \omega t)(-C_x \sin k_x x + D_x \cos k_x x). \quad (8–199)$$

Integrating, we have

$$v_x = -\frac{k_x Y Z}{\omega \rho_0} (A \sin \omega t - B \cos \omega t)(-C_x \sin k_x x + D_x \cos k_x x) \quad (8–200)$$

plus a function of $x, y, z$, which vanishes, since we are looking for oscillating solutions. In order to ensure that $v_x$ vanishes at $x = 0$, we must set $D_x = 0$, i.e., choose the cosine solution for $X$ in Eq. (8–217). This means that the pressure $p'$ must oscillate at maximum amplitude at the wall. This is perhaps obvious physically, and could have been used instead of the condition $v_x = 0$, which, however, seems more self-evident. The velocity component perpendicular to a wall must have a node at the wall, and the pressure must have an antinode. Similarly, the pressure must have an antinode (maximum amplitude of oscillation) at the wall $x = L_x$:

$$\cos k_x L_x = \pm 1, \quad (8–211)$$

so that

$$k_x = \frac{l \pi}{L_x}, \quad l = 0, 1, 2, \ldots. \quad (8–222)$$

By applying similar considerations to the four remaining walls, we conclude that $D_y = D_z = 0$, and

$$k_y = \frac{m \pi}{L_y}, \quad m = 0, 1, 2, \ldots \quad (8–223)$$

$$k_z = \frac{n \pi}{L_z}, \quad n = 0, 1, 2, \ldots.$$ 

For each choice of three integers $l, m, n$, there is a normal mode of vibra-
tion of the fluid in the box. The frequencies of the normal modes of vibration are given by Eqs. (8–216), (8–222), and (8–223):

$$\omega_{lmn} = \pi c \left( \frac{l^2}{L_x^2} + \frac{m^2}{L_y^2} + \frac{n^2}{L_z^2} \right)^{1/2}.$$  (8–224)

The three integers \(l, m, n\) cannot all be zero, for this gives \(\omega = 0\) and does not correspond to a vibration of the fluid. If we combine these results with Eqs. (8–217), (8–213), (8–211), and (8–207), we have for the normal mode of vibration characterized by the numbers \(l, m, n\):

$$p' = (A \cos \omega_{lmn} t + B \sin \omega_{lmn} t) \cos \frac{l\pi x}{L_x} \cos \frac{m\pi y}{L_y} \cos \frac{n\pi z}{L_z},$$  (8–225)

where we have suppressed the superfluous constant \(C_xC_yC_z\). The corresponding velocities are

$$v_x = \frac{l\pi}{L_x \rho_0 \omega_{lmn}} (A \sin \omega_{lmn} t - B \cos \omega_{lmn} t) \sin \frac{l\pi x}{L_x} \cos \frac{m\pi y}{L_y} \cos \frac{n\pi z}{L_z},$$

$$v_y = \frac{m\pi}{L_y \rho_0 \omega_{lmn}} (A \sin \omega_{lmn} t - B \cos \omega_{lmn} t) \cos \frac{l\pi x}{L_x} \sin \frac{m\pi y}{L_y} \cos \frac{n\pi z}{L_z},$$

$$v_z = \frac{n\pi}{L_z \rho_0 \omega_{lmn}} (A \sin \omega_{lmn} t - B \cos \omega_{lmn} t) \cos \frac{l\pi x}{L_x} \cos \frac{m\pi y}{L_y} \sin \frac{n\pi z}{L_z}.$$  (8–226)

These four equations give a complete description of the motion of the fluid for a normal mode of vibration. The walls \(x = 0, x = L_x\), and the \((l - 1)\) equally spaced parallel planes between them are nodes for \(v_x\) and antinodes for \(p', v_y\), and \(v_z\). A similar remark applies to nodal planes parallel to the other walls.

It will be observed that the normal frequencies are not, in general, harmonically related to one another, as they were in the case of the vibrating string. If, however, one of the dimensions, say \(L_x\), is much larger than the other two, so that the box becomes a long square pipe, then the lowest frequencies will correspond to the case where \(m = n = 0\) and \(l\) is a small integer, and these frequencies are harmonically related. Thus, in a pipe, the first few normal frequencies above the lowest will be multiples of the lowest frequency. This explains why it is possible to get musical tones from an organ pipe, as well as from a vibrating string. Our treatment here applies only to a closed organ pipe, and a square one at that. The treatment of a closed circular pipe is not much more difficult than the above treatment and the general nature of the results is similar. The open ended
Pipe is, however, much more difficult to treat exactly. The difficulty lies in the determination of the boundary condition at the open end; indeed, not the least of the difficulties is in deciding just where the boundary is. As a rough approximation, one may assume that the boundary is a plane surface across the end of the pipe, and that this surface is a pressure node. The results are then similar to those for the closed pipe, except that if one end of a long pipe is closed and one open, the first few frequencies above the lowest are all odd multiples of the lowest.

The general solution of the equations for sound vibrations in a rectangular cavity can be built up, as in the case of the vibrating string, by adding normal mode solutions of the form (8–225) for all normal modes of vibration. The constants $A$ and $B$ for each mode of vibration can again be chosen to fit the initial conditions, which in this case will be a specification of $p$ and $\partial p/\partial t$ (or $p'$ and $v$) at all points in the cavity at some initial instant. We shall not carry out this development here. [In the above discussion, we have omitted the case $l = m = n = 0$, which corresponds to a constant pressure increment $p'$. Likewise, we omitted steady velocity solutions $v(x, y, z)$ which do not oscillate in time. These solutions would have to be included in order to be able to fit all initial conditions.]

For cavities of other simple shapes, for example spheres and cylinders, the method of separation of variables used in the above example works, but in these cases instead of the variables $x$, $y$, $z$, coordinates appropriate to the shape of the boundary surface must be used, for example spherical or cylindrical coordinates. In most cases, except for a few simple shapes, the method of separation of variables cannot be made to work. Approximate methods can be used when the shape is very close to one of the simple shapes whose solution is known. Otherwise the only general methods of solution are numerical methods which usually involve a prohibitive amount of labor. It can be shown, however, that the general features of our results for rectangular cavities hold for all shapes; that is, there are normal modes of vibration with characteristic frequencies, and the lowest general motion is a superposition of these.

8–12 Sound waves in pipes. A problem of considerable interest is the problem of the propagation of sound waves in pipes. We shall consider a pipe whose axis is in the $z$-direction, and whose cross section is rectangular, of dimensions $L_x L_y$. This problem is the same as that of the preceding section except that there are no walls perpendicular to the $z$-axis.

We shall apply the same method of solution, the only difference being that the boundary conditions now apply only at the four walls $x = 0$, $x = L_x$, $y = 0$, $y = L_y$. Consequently, we are restricted in our choice of the functions $X(x)$ and $Y(y)$, just as in the preceding section, by Eqs. (8–217), (8–222), and (8–223). There are no restrictions on our choice of
solution of the Z-equation (8–215). Since we are interested in solutions representing the propagation of waves down the pipe, we choose the exponential form of solution for Z:

\[ Z = e^{ik_z z} \]

(8–227)

and we choose the complex exponential solution (8–212) for Θ. Our solution for \( p' \), then, for a given choice of the integers \( l, m \), is

\[ p' = \text{Re} \, A e^{i(k_z z - \omega t)} \cos \frac{l \pi x}{L_z} \cos \frac{m \pi y}{L_y} \]

\[ = A \cos \frac{l \pi x}{L_z} \cos \frac{m \pi y}{L_y} \cos (k_z z - \omega t). \]

(8–228)

This represents a harmonic wave, traveling in the \( z \)-direction down the pipe, whose amplitude varies over the cross section of the pipe according to the first two cosine factors. Each choice of integers \( l, m \) corresponds to what is called a mode of propagation for the pipe. (The choice \( l = 0, m = 0 \) is an allowed choice here.) For a given \( l, m \) and a given frequency \( \omega \), the wave number \( k_z \) is determined by Eqs. (8–216), (8–222), and (8–223):

\[ k_z = \pm \left[ \frac{\omega^2}{c^2} - \left( \frac{l \pi}{L_z} \right)^2 - \left( \frac{m \pi}{L_y} \right)^2 \right]^{1/2}. \]

(8–229)

The plus sign corresponds to a wave traveling in the \(+z\)-direction, and conversely. For \( l = m = 0 \), this is the same as the relation (8–201) for a wave traveling with velocity \( c \) in the \( z \)-direction in a fluid filling three-dimensional space. Otherwise, the wave travels with the velocity

\[ c_{lm} = \frac{\omega}{|k_z|} = c \left[ 1 - \left( \frac{l \pi c}{\omega L_z} \right)^2 - \left( \frac{m \pi c}{\omega L_y} \right)^2 \right]^{-1/2}, \]

(8–230)

which is greater than \( c \) and depends on \( \omega \). There is evidently a minimum frequency

\[ \omega_{lm} = \left[ \left( \frac{l \pi c}{L_z} \right)^2 + \left( \frac{m \pi c}{L_y} \right)^2 \right]^{1/2} \]

(8–231)

below which no propagation is possible in the \( l, m \) mode; for \( k_z \) would be imaginary, and the exponent in Eq. (8–227) would be real, so that instead of a wave propagation we would have an exponential decline in amplitude of the wave in the \( z \)-direction. Note the similarity of these results to those obtained in Section 8–4 for the discrete string, where, however, there was an upper rather than a lower limit to the frequency. Since \( c_{lm} \) depends on \( \omega \), we again have the phenomenon of dispersion. A wave of arbitrary
shape, which can be resolved into sinusoidally oscillating components of various frequencies \( \omega \), will be distorted as it travels along the pipe because each component will have a different velocity. We leave as an exercise the problem of calculating the fluid velocity \( v \), and the power flow, associated with the wave (8–228).

Similar results are obtained for pipes of other than rectangular cross section. Analogous methods and results apply to the problem of the propagation of electromagnetic waves down a wave guide. This is one reason for our interest in the present problem.

8–13 The Mach number. Suppose we wish to consider two problems in fluid flow having geometrically similar boundaries, but in which the dimensions of the boundaries, or the fluid velocity, density, or compressibility are different. For example, we may wish to investigate the flow of a fluid in two pipes having the same shape but different sizes, or we may be concerned with the flow of a fluid at different velocities through pipes of the same shape, or with the flow of fluids of different densities. We might be concerned with the relation between the behavior of an airplane and the behavior of a scale model, or with the behavior of an airplane at different altitudes, where the density of the air is different. Two such problems involving boundaries of the same shape we shall call similar problems. Under what conditions will two similar problems have similar solutions?

In order to make this question more precise, let us assume that for each problem a characteristic distance \( s_0 \) is defined which determines the geometrical scale of the problem. In the case of similar pipes, \( s_0 \) might be a diameter of the pipe. In the case of an airplane, \( s_0 \) might be the wing span. We then define dimensionless coordinates \( x', y', z' \) by the equations

\[
x' = x/s_0, \quad y' = y/s_0, \quad z' = z/s_0.
\]  

(8–232)

The boundaries for two similar problems will have identical descriptions in terms of the dimensionless coordinates \( x', y', z' \); only the characteristic distance \( s_0 \) will be different. In a similar way, let us choose a characteristic speed \( v_0 \) associated with the problem. The speed \( v_0 \) might be the average speed of flow of fluid in a pipe, or the speed of the airplane relative to the stationary air at a distance from it, or \( v_0 \) might be the maximum speed of any part of the fluid relative to the pipe or the airplane. In any case, we suppose that \( v_0 \) is so chosen that the maximum speed of any part of the fluid is not very much larger than \( v_0 \). We now define a dimensionless velocity \( v' \), and a dimensionless time coordinate \( t' \):

\[
v' = v/v_0, \quad t' = v_0 t/s_0.
\]  

(8–233)

(8–234)
We now say that two similar problems have similar solutions if the solutions are identical when expressed in terms of the dimensionless velocity \( v' \) as a function of \( x', y', z', \) and \( t' \). The fluid flow pattern will then be the same in both problems, differing only in the distance and time scales determined by \( s_0 \) and \( v_0 \). We need also to assume a characteristic density \( \rho_0 \) and pressure \( p_0 \). In the case of the airplane, these would be the density and pressure of the undisturbed atmosphere; in the case of the pipe, they might be the average density and pressure, or the density and pressure at one end of the pipe. We shall define a dimensionless pressure increment \( p'' \) as follows:

\[
p'' = \frac{p - p_0}{\rho_0 v_0^2}.
\]  

(8–235)

We shall now assume that the changes in density of the fluid are small enough so that we can write

\[
\rho = \rho_0 + \frac{d \rho}{d p} (p - p_0),
\]  

(8–236)

where higher order terms in the Taylor series for \( \rho \) have been neglected. By making use of the definition (8–235) for \( p'' \), and of the bulk modulus \( B \) as given by Eq. (5–183), this can be written

\[
\rho = \rho_0 (1 + M^2 p'') ,
\]  

(8–237)

where

\[
M = v_0 \left( \frac{B}{\rho_0} \right)^{1/2} = \frac{v_0}{c}.
\]  

(8–238)

Here \( M \) is the ratio of the characteristic velocity \( v_0 \) to the velocity of sound \( c \) and is called the *Mach number* for the problem. In a similar way, we can expand \( 1/\rho \), assuming that \( |\rho - \rho_0| \ll \rho_0 \):

\[
\frac{1}{\rho} = \frac{1}{\rho_0} (1 - M^2 p'').
\]  

(8–239)

With the help of Eqs. (8–237) and (8–239), we can rewrite the equation of continuity and the equation of motion in terms of the dimensionless variables introduced by Eqs. (8–232) to (8–235). The equation of continuity (8–127), when we divide through by the constant \( \rho_0 v_0/s_0 \) and collect separately the terms involving \( M \), becomes

\[
\nabla' \cdot \nabla' + M^2 \left[ \frac{\partial p''}{\partial v'} + \nabla' \cdot (p'' \nabla') \right] = 0,
\]  

(8–240)

where

\[
\nabla' = i \frac{\delta}{\delta x'} + j \frac{\delta}{\delta y'} + k \frac{\delta}{\delta z'}.
\]  

(8–241)
The equation of motion (8–139), when we divide through by \( v_0^2 / s_0 \), becomes, in the same way,

\[
\frac{\partial v'}{\partial t'} + v' \cdot \nabla v' + (1 - M'^2 \rho') \nabla \rho' = \frac{s_0}{v_0^2} \frac{f}{\rho}.
\]  

Equations (8–240) and (8–242) represent four differential equations to be solved for the four quantities \( p' \), \( v' \), subject to given initial and boundary conditions. If the body forces are zero, or if the body forces per unit mass \( f/\rho \) are made proportional to \( v_0^2 / s_0 \), then the equations for two similar problems become identical if the Mach number \( M \) is the same for both. Hence, similar problems will have similar solutions if they have the same Mach number. Results of experiments on scale models in wind tunnels can be extrapolated to full-sized airplanes flying at speeds with corresponding Mach numbers. If the Mach number is much less than one, the terms in \( M'^2 \) in Eqs. (8–240) and (8–242) can be neglected, and these equations then reduce to the equations for an incompressible fluid, as is obvious either from Eq. (8–240) or (8–237). Therefore at fluid velocities much less than the speed of sound, even air may be treated as an incompressible fluid. On the other hand, at Mach numbers near or greater than one, the compressibility becomes important, even in problems of liquid flow. Note that the Mach number involves only the characteristic velocity \( v_0 \), and the velocity of sound, which in turn depends on the characteristic density \( \rho_0 \) and the compressibility \( B \). Changes in the distance scale factor \( s_0 \) have no effect on the nature of the solution, nor do changes in the characteristic pressure \( p_0 \) except insofar as they affect \( \rho_0 \) and \( B \).

It must be emphasized that these results are applicable only to ideal fluids, i.e., when viscosity is unimportant, and to problems where the density of the fluid does not differ greatly at any point from the characteristic density \( \rho_0 \). The latter condition holds fairly well for liquids, except when there is cavitation (formation of vapor bubbles), and for gases except at very large Mach numbers.

8–14 Viscosity. In many practical applications of the theory of fluid flow, it is not permissible to neglect viscous friction, as has been done in the preceding sections. When adjacent layers of fluid are moving past one another, this motion is resisted by a shearing force which tends to reduce their relative velocity. Let us assume that in a given region the velocity of the fluid is in the \( x \)-direction, and that the fluid is flowing in layers parallel to the \( xz \)-plane, so that \( v_x \) is a function of \( y \) only (Fig. 8–10). Let the positive \( y \)-axis be directed toward the right. Then if \( \partial v_x / \partial y \) is positive, the viscous friction will result in a positive shearing force \( F_x \) acting from right to left across an area \( A \) parallel to the \( xz \)-plane. The
The coefficient of viscosity \( \eta \) is defined as the ratio of the shearing stress to the velocity gradient:

\[
\eta = \frac{F_z}{A} \frac{\partial u_z}{\partial y}.
\] (8–243)

When the velocity distribution is not of this simple type, the stresses due to viscosity are more complicated. (See Section 10–6.)

We shall apply this definition to the important special case of steady flow of a fluid through a pipe of circular cross section, with radius \( a \). We shall assume laminar flow; that is, we shall assume that the fluid flows in layers, as contemplated in the definition above. In this case, the layers are cylinders. The velocity is everywhere parallel to the axis of the pipe, which we take to be the \( z \)-axis, and the velocity \( u_z \) is a function only of \( r \), the distance from the axis of the pipe. (See Fig. 8–11.) If we consider a cylinder of radius \( r \) and of length \( l \), its area will be \( A = 2\pi rl \), and according to the definition (8–243), the force exerted across this cylinder by the fluid outside on the fluid inside the cylinder is

\[
F_z = \eta (2\pi rl) \frac{du_z}{dr}.
\] (8–244)

Since the fluid within this cylinder is not accelerated, if there is no body force the viscous force must be balanced by a difference in pressure
between the two ends of the cylinder:

\[ \Delta p (\pi r^2) + F_z = 0, \quad (8-245) \]

where \( \Delta p \) is the difference in pressure between the two ends of the cylinder a distance \( l \) apart, and we assume that the pressure is uniform over the cross section of the pipe. Equations (8-244) and (8-245) can be combined to give a differential equation for \( v_z \):

\[ \frac{dv_z}{dr} = -\frac{r \Delta p}{2 \eta l}. \quad (8-246) \]

We integrate outward from the cylinder axis:

\[ \int_{v_0}^{v_z} dv_z = -\frac{\Delta p}{2 \eta l} \int_{0}^{r} r \, dr, \]

\[ v_z = v_0 - \frac{r^2 \Delta p}{4 \eta l}, \quad (8-247) \]

where \( v_0 \) is the velocity at the axis of the pipe. We shall assume that the fluid velocity is zero at the walls of the pipe:

\[ [v_z]_{r=a} = v_0 - \frac{a^2 \Delta p}{4 \eta l} = 0, \quad (8-248) \]

although this assumption is open to question.
Then
\[ v_0 = \frac{a^2 \Delta p}{4\eta l}, \] (8–249)
and
\[ v_z = \frac{\Delta p}{4\eta l} (a^2 - r^2). \] (8–250)

The total fluid current through the pipe is
\[ I = \iiint \rho v_z \, dS = 2\pi \rho \int_0^a v_z r \, dr. \] (8–251)

We substitute from Eq. (8–250) and carry out the integration:
\[ \frac{I}{\rho} = \frac{\pi a^4 \Delta p}{8\eta l}. \] (8–252)

This formula is called Poiseuille's law. It affords a convenient and simple way of measuring \( \eta \).

Although we will not develop now the general equations of motion for viscous flow, we can arrive at a result analogous to that in Section 8–13, taking viscosity into account, without actually setting up the equations for viscous flow. Suppose that we are concerned, as in Section 8–13, with two similar problems in fluid flow, and let \( s_0, v_0, \rho_0, \rho_0 \) be a characteristic distance, velocity, pressure, and density, which again define the scale in any problem. However, let us suppose that in this case viscosity is to be taken into account, so that the equation of motion (8–139) is augmented by a term corresponding to the force of viscous friction. We do not at present know the precise form of this term, but at any rate it will consist of \( \eta \) multiplied by various derivatives of various velocity components, and divided by \( \rho \) [since Eq. (8–139) has already been divided through by \( \rho \)]. When we introduce the velocity \( \mathbf{v}' \), and the dimensionless coordinates \( x', y', z', t' \), as in Section 8–13, and divide the equation of motion by \( v_0^2/s_0 \), we will obtain just Eq. (8–242), augmented by a term involving the coefficient of viscosity. Since all the terms in Eq. (8–242) are dimensionless, the viscosity term will be also, and will consist of derivatives of components of \( \mathbf{v}' \) with respect to \( x', y', z' \), multiplied by numerical factors and by a dimensionless coefficient consisting of \( \eta \) times some combination of \( v_0 \) and \( s_0 \), and divided by \( \rho = \rho_0(1 + M^2p'') \) [Eq. (8–237)]. Now the dimensions of \( \eta \), as determined by Eq. (8–243), are
\[ [\eta] = \frac{\text{mass}}{\text{length} \times \text{time}}, \] (8–253)
and the only combination of \( \rho_0, v_0, \) and \( s_0 \) having these dimensions
is \( \rho_0v_0s_0 \). Therefore the viscosity term will be multiplied by the coefficient

\[
\frac{1}{R(1 + M^2p''')} ,
\]

(8–254)

where \( R \) is the Reynolds number, defined by

\[
R = \frac{\rho_0v_0s_0}{\eta} .
\]

(8–255)

We can now conclude that when viscosity is important, two similar problems will have the same equation of motion in dimensionless variables, and hence similar solutions, only if the Reynolds number \( R \), as well as the Mach number \( M \), is the same for both. If the Mach number is very small, then compressibility is unimportant. If the Reynolds number is very large, then viscosity may be neglected. It turns out that there is a critical value of Reynolds number for any given problem, such that the nature of the flow is very different for \( R \) larger than this critical value than for smaller values of \( R \). For small Reynolds numbers, the flow is laminar, as the viscosity tends to damp out any vortices which might form. For large Reynolds numbers, the flow tends to be turbulent. This will be the case when the viscosity is small, or the density, velocity, or linear dimensions are large. Note that the Reynolds number depends on \( s_0 \), whereas the Mach number does not, so that the distance scale of a problem is important when the effects of viscosity are considered. Viscous effects are more important on a small scale than on a large scale.

It may be noted that the expression (8–255) for the Reynolds number, together with the fact that Eq. (8–139) is divided by \( v_0^2/s_0 \) to obtain the dimensionless equation of motion, implies that the viscosity term to be added to Eq. (8–139) has the dimensions of \( (\eta v_0)/(\rho_0 s_0^2) \). This, in turn, implies that the viscous force density must be equal to \( \eta \) times a sum of second derivatives of velocity components with respect to \( x \), \( y \), and \( z \). This is perhaps also evident from Eq. (8–243), since in calculating the total force on a fluid element, the differences in stresses on opposite faces of the element will be involved, and hence a second differentiation of velocities relative to \( x \), \( y \), and \( z \) will appear in the expression for the force. An expression for the viscous force density will be developed in Chapter 10.