II. Generalizing the 1-dimensional wave equation

First generalize the notation.

i) "q" has meant transverse deflection of the string.

Replace $q \rightarrow \Psi$, where $\Psi$ may indicate other properties of the medium that depend on position and time. This is a capital letter $\Psi$.

$\Psi$, the "wave function," could also be

- longitudinal deflection
- magnitude of an electric field
- quantum mechanical probability amplitude

...

ii) Note the dimensions of $\frac{\rho}{\tau}$ are $\frac{kg}{sec^2}$

$$\frac{kg}{sec^2} = \frac{sec^2}{meter^2} = \frac{1}{velocity^2}$$

We will interpret this velocity in upcoming slides.
For now, we substitute these generalizations into the 1-dimensional wave equation to obtain:

\[ \frac{\partial^2 \Psi}{dx^2} - \frac{1}{v^2} \frac{\partial^2 \Psi}{dt^2} = 0 \]

"The general one-dimensional wave equation"
I. Solving the wave equation
II. Wave number
III. Wave propagation
IV. Functions $f(x + vt)$ and $g(x - vt)$
I. Solving the wave equation...
...will help us better understand what a wave is.

Recall \( \frac{\partial^2 \Psi}{dx^2} - \frac{1}{v^2} \frac{\partial^2 \Psi}{dt^2} = 0 \)

Guess \( \Psi \) has the separable form:
\[
\Psi(x,t) = \psi(x) \cdot \chi(t).
\]

Then
\[
\frac{\partial^2 \Psi}{dx^2} = \frac{\partial^2 \psi}{dx^2} \chi
\]
\[
\frac{\partial^2 \Psi}{dt^2} = \psi \frac{\partial^2 \chi}{dt^2}
\]

The wave equation becomes
\[
\frac{\partial^2 \psi}{dx^2} \chi - \frac{1}{v^2} \psi \frac{\partial^2 \chi}{dt^2} = 0
\]
\[
\frac{\partial^2 \psi(x)}{dx^2} \cdot \frac{v^2}{\psi(x)} = \frac{\partial^2 \chi(t)}{dt^2} \cdot \frac{1}{\chi(t)}
\]

Function of \(x\) only = Function of \(t\) only

This equation can be true only if both sides actually equal a constant that is neither a function of \(x\) nor a function of \(t\).

Name that constant "\(-\omega^2\)".

\[
\begin{align*}
\text{LHS} & \quad \frac{\partial^2 \psi(x)}{dx^2} \cdot \frac{v^2}{\psi(x)} = -\omega^2 \\
& \quad \frac{\partial^2 \psi}{dx^2} = -\frac{\omega^2 \psi}{v^2} \\
& \quad \frac{\partial^2 \psi}{dx^2} + \frac{\omega^2 \psi}{v^2} = 0 \\
\text{RHS} & \quad \frac{\partial^2 \chi(t)}{dt^2} \cdot \frac{1}{\chi(t)} = -\omega^2 \\
& \quad \frac{\partial^2 \chi}{dt^2} + \omega^2 \chi = 0
\end{align*}
\]

\[
\psi(x) = Ae^{i(\omega \frac{t}{v})} + Be^{-i(\omega \frac{t}{v})} \\
\chi(t) = Ce^{i\omega t} + De^{-i\omega t}
\]
So $\Psi = \psi(x) \chi(t) = A \cdot Ce^{i\left(\frac{\omega}{v}x + \omega t\right)} + B \cdot Ce^{-i\left(\frac{\omega}{v}x - \omega t\right)} + A \cdot De^{i\left(\frac{\omega}{v}x - \omega t\right)} + B \cdot De^{-i\left(\frac{\omega}{v}x + \omega t\right)}$

Notice:

This equation represents propagation of a wave with a single frequency $\omega$. But we know from Chapter 12 that a system with $n$ degrees of freedom (for example a string loaded with $n$ masses) oscillates with $n$ normal modes, and so potentially with $n$ different frequencies. Call those frequencies the $\omega_r$, $(r = 1, 2, \ldots n)$. Call the associated normal modes $\Psi_r$.

Then $\Psi_{tot} = \sum_r \Psi_r = \sum_r \left[ a_re^{i\left(\frac{\omega_r}{v}x + \omega_rt\right)} + b_re^{-i\left(\frac{\omega_r}{v}x - \omega_rt\right)} + c_re^{i\left(\frac{\omega_r}{v}x - \omega_rt\right)} + d_re^{-i\left(\frac{\omega_r}{v}x + \omega_rt\right)} \right]$

where the $a_r$ are a family of amplitudes like $A \cdot C$, one for each normal mode.
II. Wave number

To understand the characteristics of $\Psi_{tot}$, we study just one representative term:

$$\Psi_r \sim \exp \left[ -i \left( \frac{\omega}{v} \right) (x - vt) \right].$$

For simplicity we now drop the $r$-subscript.

$$\Psi \sim \exp \left[ -i \left( \frac{\omega}{v} \right) (x - vt) \right].$$

Define $\frac{\omega}{v} \equiv k$, the "wave number".

Then this representative wave is given by:

$$\Psi \sim \exp \left[ -i k (x - vt) \right] = e^{+i(\omega t - kx)}.$$

For any wave, velocity $= \text{wavelength} \times \text{frequency}$

$$v = \lambda \nu$$

$$\frac{\text{meters}}{\text{sec}} = \text{meters} \cdot \frac{1}{\text{sec}}$$
But frequency $= \frac{\text{angular frequency}}{2\pi}$

cycles $= \frac{\text{radians}}{\text{sec}} = \frac{\text{sec}}{2\pi \text{ radians}}$

cycle $= \frac{\omega}{2\pi}$

So $v = \lambda \cdot \frac{\omega}{2\pi}$

Rearrange:

$\frac{2\pi}{\lambda} = \frac{\omega}{v} = k$

Emphasize: $k = \frac{2\pi}{\lambda}$
III. Wave propagation

Goal: to see that the equation, \( \frac{\partial^2 \Psi}{dx^2} - \frac{1}{v^2} \frac{\partial^2 \Psi}{dt^2} = 0 \), is solved by ANY functions of the form \( f(x + vt) \) and \( g(x - vt) \) that propagate with constant shape and velocity \( v \). That is: functions whose shape remains constant as \( t \) increases, because -

– for \( f \), as \( t \) increases, \( x \) decreases at a rate that maintains \( (x + vt) \), i.e., \( f \) propagates to the LEFT.

The \textit{x decreasing} implies LEFTWARD propagation.

– for \( g \), as \( t \) increases, \( x \) increases at a rate that maintains \( (x - vt) \), i.e., \( g \) propagates to the RIGHT.

The \textit{x increasing} implies RIGHTWARD propagation.
Note that $f$ and $g$ can have any shape. They do not have to be harmonic or even periodic.

Example shapes:

Vocabulary: "$x \pm vt$" is called the phase of the wave.
We now prove that any function of the form \( f(x + vt) \) or \( g(x - vt) \) solves the one-dimensional wave equation.

Define \( \xi \equiv x + vt \)

and \( \eta \equiv x - vt \)

Our goal is to write the wave equation,

\[
\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2}
\]

'Eq 1'

in terms of these variables.

We need to calculate a lot of derivatives.

\[
\frac{\partial \Psi}{\partial x} = \frac{\partial \Psi}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \Psi}{\partial \eta} \frac{\partial \eta}{\partial x}
\]

Thus:

\[
\frac{\partial \Psi}{\partial x} = \frac{\partial \Psi}{\partial \xi} + \frac{\partial \Psi}{\partial \eta}
\]
\[
\frac{\partial^2 \Psi}{dx^2} = \frac{\partial}{\partial x} \left( \frac{\partial \Psi}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial \Psi}{\partial \xi} + \frac{\partial \Psi}{\partial \eta} \right)
\]

\[
= \frac{\partial}{\partial \xi} \left( \frac{\partial \Psi}{\partial \xi} + \frac{\partial \Psi}{\partial \eta} \right) \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta} \left( \frac{\partial \Psi}{\partial \xi} + \frac{\partial \Psi}{\partial \eta} \right) \frac{\partial \eta}{\partial x}
\]

\[
\begin{array}{c}
1 \\
1
\end{array}
\]

\[
= \frac{\partial^2 \Psi}{d\xi^2} + \frac{\partial^2 \Psi}{\partial \xi \partial \eta} + \frac{\partial^2 \Psi}{\partial \eta \partial \xi} + \frac{\partial^2 \Psi}{\partial \eta^2}
\]

But these are scalars, so the order of differentiation \( \frac{\partial^2}{\partial \xi \partial \eta} \) versus \( \frac{\partial^2}{\partial \eta \partial \xi} \) is unimportant. So:

\[
\frac{\partial^2 \Psi}{dx^2} = \frac{\partial^2 \Psi}{d\xi^2} + 2 \frac{\partial^2 \Psi}{\partial \xi \partial \eta} + \frac{\partial^2 \Psi}{\partial \eta^2}
\]

We need this for the LHS of Eq 1.
Similarly, \[ \frac{\partial \Psi}{\partial t} = \frac{\partial \Psi}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial \Psi}{\partial \eta} \frac{\partial \eta}{\partial t} \]

\[
\begin{align*}
\frac{\partial \Psi}{\partial t} &= \nu \left[ \frac{\partial \Psi}{\partial \xi} - \frac{\partial \Psi}{\partial \eta} \right] \\
\frac{\partial^2 \Psi}{\partial t^2} &= \frac{\partial}{\partial t} \left\{ \nu \left[ \frac{\partial \Psi}{\partial \xi} - \frac{\partial \Psi}{\partial \eta} \right] \right\} \\
&= \nu \left\{ \frac{\partial}{\partial \xi} \left[ \frac{\partial \Psi}{\partial \xi} - \frac{\partial \Psi}{\partial \eta} \right] \frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta} \left[ \frac{\partial \Psi}{\partial \xi} - \frac{\partial \Psi}{\partial \eta} \right] \frac{\partial \eta}{\partial t} \right\}
\end{align*}
\]

So \[ \frac{1}{\nu^2} \frac{\partial^2 \Psi}{\partial t^2} = \frac{\partial^2 \Psi}{\partial \xi^2} - 2 \frac{\partial^2 \Psi}{\partial \xi \partial \eta} + \frac{\partial^2 \Psi}{\partial \eta^2} \]

This is the RHS of Eq 1.
Substitute into the LHS and RHS of Eq 1 to get:

\[
\frac{\partial^2 \Psi}{d \xi^2} + 2 \frac{\partial^2 \Psi}{\partial \xi \partial \eta} + \frac{\partial^2 \Psi}{\partial \eta^2} = \frac{\partial^2 \Psi}{d \xi^2} - 2 \frac{\partial^2 \Psi}{\partial \xi \partial \eta} + \frac{\partial^2 \Psi}{\partial \eta^2}
\]

This can only be true if \(2 \frac{\partial^2 \Psi}{\partial \xi \partial \eta} = -2 \frac{\partial^2 \Psi}{\partial \xi \partial \eta}\).

That is, if \(\frac{\partial^2 \Psi}{\partial \xi \partial \eta} = 0\), 'linearity condition'

i.e., if \(\Psi\) is not a function of \(\xi \cdot \eta\).

The most general solution of the linearity condition equation is:

\[\Psi = f(\xi) + g(\eta)\]

Thus:

\[\Psi = f(x + vt) + g(x - vt)\] for arbitrary \(f\) and \(g\).
IV. What are \( f \) and \( g \)?

Recall \( \Psi = f(x+vt) + g(x-vt) \)

\[ \xi \quad \eta \]

Suppose the initial position of every point on the string is

\[ \Psi(x,0) = f(x) + g(x) \equiv \Psi_0(x) \quad \text{'Eq 1'} \]

Similarly the initial velocities are

\[
\left. \frac{\partial \Psi}{\partial t} \right|_{t=0} = \left[ \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial g}{\partial \eta} \frac{\partial \eta}{\partial t} \right]_{t=0} = \left[ v \frac{\partial f}{\partial \xi} - v \frac{\partial g}{\partial \eta} \right]_{t=0} \equiv v_0(x) \quad \text{'Eq 2'}
\]

Remember that \( \Psi \) is a generalization of displacement \( q \), so \( \frac{\partial \Psi}{\partial t} \) is a velocity.
When $t = 0$, $\xi = \eta = x$.

Then Eq 2 becomes

$$v \cdot \frac{d}{dx} \left[ f(x) - g(x) \right] \equiv v_0(x)$$

$$\frac{d}{dx} \left[ f(x) - g(x) \right] = \frac{v_0(x)}{v}$$

Integrate:

$$f(x) - g(x) = \frac{1}{v} \int_0^x v_0(x') dx' \quad \text{Eq 3'}$$

Solve Eq 1 and Eq 3 simultaneously to extract $f$ and $g$:

Eq 1 - Eq 3:

$$2g(x) = \Psi_0(x) - \frac{1}{v} \int_0^x v_0(x') dx'$$

$$g(x) = \frac{1}{2} \left[ \Psi_0(x) - \frac{1}{v} \int_0^x v_0(x') dx' \right]$$
Eq 1 + Eq 3:

\[ 2f(x) = \Psi_0(x) + \frac{1}{v} \int_0^x v_0(x') \, dx' \]

\[ f(x) = \frac{1}{2} \left[ \Psi_0(x) + \frac{1}{v} \int_0^x v_0(x') \, dx' \right] \]

Note: These equations are valid for all times (we are free to label ANY time "t = 0") so we can now replace \( x \rightarrow \xi \) or \( \eta \) as appropriate. Then

\[ f(\xi) = \frac{1}{2} \left[ \Psi_0(\xi) + \frac{1}{v} \int_0^\xi v_0(\xi') \, d\xi' \right] \]

\[ g(\eta) = \frac{1}{2} \left[ \Psi_0(\eta) - \frac{1}{v} \int_0^\eta v_0(\eta') \, d\eta' \right] \]

Message: a wave is specified \textbf{completely} by its initial conditions.
I. Driven and damped waves
II. Wave energy flow and boundary conditions
III. Standing waves
I. Driven and damped waves
Recall the 1-dimensional wave equation
\[
\frac{\partial^2 q}{dx^2} - \frac{\rho}{\tau} \frac{\partial^2 q}{dt^2} = 0
\]
was derived for a medium responding only to restoring ("spring") forces.

Recall \(\rho\) is linear mass density (mass per length) and \(\tau\) is string tension which has units of force.

Our goal now: to incorporate driving and damping forces too.
Multiply through by string tension \(\tau\):
\[
\tau \frac{\partial^2 q}{dx^2} - \rho \frac{\partial^2 q}{dt^2} = 0
\]
Both terms have units of force per length.
Rewrite to match the form of Thornton Eq 13.39.

\[ \rho \frac{\partial^2 q}{dt^2} - \tau \frac{\partial^2 q}{dx^2} = 0 \]

Add to this:

- a driving force \( F(x,t) \) acting along the string, with units of force per length.
- a damping force per length, \( D \frac{\partial q}{dt} \) (like friction), where \( D \) has units \( \frac{kg}{m \cdot s} \).

\[ \rho \frac{\partial^2 q}{dt^2} + D \frac{\partial q}{dt} - \tau \frac{\partial^2 q}{dx^2} = F(x,t) \]

To solve this, guess [as in Slide 321]:

\[ q(x,t) = \sum_{N=1}^{\infty} \eta_N(t) \sin\left( \frac{N\pi x}{L} \right) \]

where \( \eta \) are normal coordinates, \( N \) are normal modes, \( L \) is string length.

This yields:

\[ \sum_{N=1}^{\infty} \left[ \left( \rho \ddot{\eta}_N + D\dot{\eta}_N + \frac{N^2 \pi^2 \tau}{L^2} \eta_N \right) \sin\left( \frac{N\pi x}{L} \right) \right] = F(x,t) \]
Multiply both sides by \( \sin\left(\frac{s \pi x}{L}\right) \), integrate both sides over \( \int_0^L dx \), and [recall Slide 316] the orthogonality of the sine function produces:

\[
\sum_{N=1}^{\infty} \left[ \left( \rho \ddot{\eta}_N + D \dot{\eta}_N + \frac{N^2 \pi^2 \tau}{L^2} \eta_N \right) \frac{L}{2} \delta_{s,N} \right] = \int_0^L F(x,t) \sin\left(\frac{s \pi x}{L}\right) dx
\]

Do the sum. Only the \( N = s \) term survives.

\[
\left( \rho \ddot{\eta}_N + D \dot{\eta}_N + \frac{N^2 \pi^2 \tau}{L^2} \eta_N \right) \frac{L}{2} = \int_0^L F(x,t) \sin\left(\frac{s \pi x}{L}\right) dx
\]

\[
\ddot{\eta}_N + \frac{D}{\rho} \dot{\eta}_N + \frac{N^2 \pi^2 \tau}{\rho L^2} \eta_N = 2 \frac{L}{\rho L} \int_0^L F(x,t) \sin\left(\frac{s \pi x}{L}\right) dx
\]

The particular solution can be obtained with standard techniques for ODE's, see for example Thornton Appendix C.2 and Thornton Eq. 3.53.
Note that driving forces can be incorporated into the general matrix-based solution technique [slide 295] that looks like \( \mathbf{A} \mathbf{q} + \mathbf{M} \ddot{\mathbf{q}} = 0 \).

If the driving forces can be written as a potential \( U' = -\sum_{k} x_k F_k(t) \), build them into the \( \mathbf{A} \) matrix along with the spring potential.

Damping forces cannot generally be solved with this technique. We would expect them to produce a matrix equation of the form \( \mathbf{A} \mathbf{q} + \mathbf{D} \mathbf{q} + \mathbf{M} \ddot{\mathbf{q}} = 0 \).

It is not generally possible simultaneously to diagonalize all three tensors \( \mathbf{A}, \mathbf{D}, \text{and} \mathbf{M} \).
II. Wave energy flow and boundary conditions

3 steps to this section:

Step 1: obtain the wave equation from $\vec{F} = m\ddot{a}$. Identify the force $\vec{F}$.

Step 2: Use this force $\vec{F}$ to compute energy flow along the medium, noting that energy flow is power per time, and for velocity $\vec{v}$, power $= \vec{F} \cdot \vec{v}$.

Step 3: Use the power formula to understand boundary conditions.

Step 1:
Consider the string in the process of oscillating with small amplitudes. The string is fixed at $x = 0$ and $L$.

Its deflection at $x$ is $q$.

Its deflection at $(x + dx)$ is $(q + dq)$.
Consider a segment of length $dx$.

Let $\rho = \frac{\text{mass}}{\text{length}} = \text{mass density}

The mass of that segment is $dm = \rho dx$. \hspace{1cm} \text{'Eq 1'}$

The string velocity (up or down) is $V = \frac{dq}{dt}$,

so the string acceleration is $a = \frac{dV}{dt} = \frac{d}{dt} \left( \frac{dq}{dt} \right) = \frac{d^2 q}{dt^2} \hspace{1cm} \text{'Eq 2'}$

The slope of the segment is $\frac{dq}{dx}$.

The segment makes angle $\theta$ with its equilibrium position.

String tension = $\tau$.

When the segment is moving up, this is because there is a vertical component to the tension. For non-zero slope, each segment is pulled up by its neighboring elements.
The vertical component of the tension is

\[ \tau_v = \tau \sin \theta \]

For small deflections, \( \sin \theta \approx \tan \theta = \frac{dq}{dx} \), so

\[ \tau_v \approx \tau \frac{dq}{dx} \]

The net force \( dF \) on the segment, due to tension, is the difference in tension of the segment ends:

\[ dF = [\tau_v]_{x+dx} - [\tau_v]_x \]

\[ dF = \frac{d}{dx} \left( \tau \frac{dq}{dx} \right) dx \]

Suppose \( \tau \neq \tau(x) \).

\[ dF = \tau \frac{d^2q}{dx^2} dx \]

'Eq 3'
Plug into $F = ma$:

$$dF = dm \cdot a$$

(Eq 3) (Eq 1) (Eq 2)

$$\tau \frac{d^2q}{dx^2} dx = \rho dx \frac{dq^2}{dt^2}$$

$$\frac{d^2q}{dx^2} - \frac{\rho}{\tau} \frac{dq^2}{dt^2} = 0$$

Recall again [Slide 328] that $\frac{\rho}{\tau}$ has units $\frac{1}{v^2}$. 

Step 2:
Recall power $P = \vec{F} \cdot \vec{v}$.
Consider a rightward travelling wave $g(x - vt) = g(\eta)$.
It propagates due to upward force exerted by the left half of the string upon the right half. We expect a rightward flow of power.
Upward velocity of the string: $\frac{\partial \Psi}{\partial t}$.
Upward force: $-\tau \frac{\partial \Psi}{\partial x}$

So $P = -\tau \frac{\partial \Psi}{\partial x} \frac{\partial \Psi}{\partial t}$

minus sign because positive string slope indicates downward force.
\[ P = -\tau \frac{\partial \Psi}{\partial x} \frac{\partial \Psi}{\partial t} \]
\[ = -\tau \left[ \frac{\partial \Psi}{\partial \eta} \frac{\partial \eta}{\partial x} \right] \left[ \frac{\partial \Psi}{\partial \eta} \frac{\partial \eta}{\partial t} \right] \]
\[ = -\tau \left[ \frac{\partial \Psi}{\partial \eta} \cdot 1 \right] \left[ \frac{\partial \Psi}{\partial \eta} \cdot (-v) \right] \]
\[ = \nu \tau \left( \frac{\partial \Psi}{\partial \eta} \right)^2 \quad \text{which is positive: rightward} \]

Similarly, for \( \Psi = f(x + vt) \) \quad \text{(leftward)}

Power \( P \) is negative \quad \text{(leftward)}

For use later:

If \( \Psi = A \cos(kx - \omega t) \),
\[ P = k\omega \tau A^2 \sin^2(kx - \omega t) \]

Then time-averaged power \( \langle P \rangle = \frac{k}{2} \omega \tau A^2 \)
Step 3: Understanding the boundary conditions.

For a string tied at $x = 0$ and $L$, boundary conditions are
\[ \Psi(x = 0) = 0 \]
\[ \Psi(x = L) = 0 \]
\[ \frac{\partial \Psi}{\partial t}(x = 0) = 0 \]
\[ \frac{\partial \Psi}{\partial t}(x = L) = 0 \]

But what boundary conditions apply if the string is free to oscillate at one end? The power formula helps us decide.
For a string terminating at \( x = L \) but free to oscillate there ("tied to a ring of negligible mass on a frictionless vertical rod"), we find the boundary conditions by noting:

At \( x = L \), no power can be transmitted to the right, so

\[
P = -\tau \frac{\partial \Psi}{\partial x} \frac{\partial \Psi}{\partial t} = 0.
\]

But since the ring is free to move up and down,

\[
\frac{\partial \Psi}{\partial t} \neq 0.
\]

Conclude:

\[
\frac{\partial \Psi}{\partial x}(x = L) = 0 \text{ but there is NO requirement that } \Psi(x = L) = 0
\]
III. Standing waves

Recall $\Psi = A_1 e^{i(kx+\omega t)} + A_2 e^{-i(kx-\omega t)} + A_3 e^{i(kx-\omega t)} + A_4 e^{-i(kx+\omega t)}$

Suppose $A_2 = A_4 = A$

and $A_1 = A_3 = 0$

Then $\Psi = Ae^{-i(kx-\omega t)} + Ae^{-i(kx+\omega t)}$

$$= Ae^{-ikx} (e^{-i\omega t} + e^{+i\omega t})$$

$$= Ae^{-ikx} \cdot 2 \cos \omega t$$

The physically meaningful solution is the real part of this complex $\Psi$.

$$\Psi_{\text{physical}} = 2A \cos \omega t \Re(\cos kx - i \sin kx)$$

$$= 2A \cos \omega t \cos kx$$

This wave does not travel. For a given $x$, $\cos kx$ is a fixed amplitude. As time $t$ increases, $\cos \omega t$ goes through its cycle, but $x$ and $t$ are not coupled, so propagation of the pattern does not occur.
This $\Psi_{\text{physical}}$ is called a standing wave.

Note that for any integer $n$,

if $x = \frac{(2n+1)\pi}{2k}$,

then $\cos kx = \cos \left[ \frac{(2n+1)\pi}{2} \right]$

$= \cos \left[ \left( n + \frac{1}{2} \right)\pi \right]$

$= 0$.

These $x$ locations at which there is NEVER oscillation are called nodes of the standing wave.
I. Reflection

II. Wave Continuity

III. Phase velocity

IV. Dispersion

Please read "Reading assignment on statics and fluids" linked to the course webpage.
I. Wave reflection

Combining the fact that $\Psi = f_{\text{leftward}} + g_{\text{rightward}}$ with a boundary condition, to predict inversion of the reflected wave.

Recall that any wave can be constructed from a superposition of leftward and rightward travelling functions:

$\Psi = f(x + vt) + g(x - vt)$

Again let $\xi = x + vt$

and $\eta = x - vt$

$\Psi = f(\xi) + g(\eta)$.

At $x = 0$, the string is tied and cannot oscillate, so demand:

$\Psi(x = 0) = 0$.

$f(0 + vt) + g(0 - vt) = 0$

$f(vt) = -g(-vt)$
Notice that for this special value of $x$, $\xi = -\eta$.
So the boundary condition is, for any time $t$,
\[ f(\xi) = -g(\eta) \text{ when } \xi = -\eta. \]  
'Eq 1'

Consider a purely right-ward travelling wave, $g(x - vt)$, travelling from $-\infty$ toward 0.
At time $t_0$, it has phase $\eta_0$ and amplitude $g(\eta_0)$ at location
\[ x_0 = \eta_0 + vt_0. \]  
'Eq 2'
where $x_0$ is a negative number.

\[ g(\eta_0) \]
\[ x_0 \quad x=0 \quad t = 0 \]
At a later time \( t_1 \), the same phase \( \eta_0 \) will have moved rightward, to
\[ x_1 = \eta_0 + \nu t_1. \]
But also, \( x_1 = x_0 + \nu (t_1 - t_0) \).
In particular, \( x_1 = 0 \) when
\[ x_0 + \nu (t_1 - t_0) = 0, \]
which occurs at
\[ t_1 = t_0 - \frac{x_0}{\nu}. \]
At later times, $x_1 > 0$, so the rightward travelling wave has no physical meaning. But at $x = 0$, the full solution for $\Psi$ allows for the existence of a leftward-travelling wave.

It has phase $\xi_0 = -\eta_0$ at $x = 0$, which occurs when $t = t_1$. So its amplitude there is $f(\xi_0) = -g(\eta_0)$ from the boundary condition, Equation 1. Notice: the amplitude is negative.

\[
\Psi = 0 \quad (amplitude)
\]

\[
\begin{aligned}
&f \\
&x = 0 \\
&t = t_1
\end{aligned}
\]
At time \( t_1 \), this phase \( \xi_0 \) will be at location \( x_2 = \xi_0 - vt_1 \).

\[
\begin{align*}
  t &= t_1 \\
  x_2 &
\end{align*}
\]

But \( \xi_0 = -\eta_0 = -(x_0 - vt_0) \) \( \quad \text{from Eq 2.} \)

So \( x_2 = -(x_0 - vt_0) - vt_1 \)

\[
- x_0 - v(t_1 - t_0)
\]

Since \( t_i > t_0 \), \( x_2 \) is negative, so \( f \) is propagating leftward.
II. Wave continuity
This section is about the relationship among incidence, reflection, and transmission.

In any region of \((x, t)\) space, the most general expression for a wavefunction is \(\Psi = f(x - vt) + g(x + vt)\). If that wave has a single frequency \(\omega\), then it must be harmonic. (More complicated waveforms must be constructed from Fourier series with multiple frequencies.) If its wave number is \(k\), we can choose to write it as \(\Psi = Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)}\).

If that wave encounters any change of environment ("boundary"), it may be reflected or transmitted. Here is the procedure to determine what happens.
**Procedure**

0) Identify all regions where a mathematical description of the wave is possible.

1) Assume the most general form of wave in each region.

\[ \Psi_I = Ae^{i(\omega t - k_1 x)} + Be^{i(\omega t + k_1 x)} \]

\[ \Psi_{II} = Ce^{i(\omega t - k_2 x)} + De^{i(\omega t + k_2 x)} \]

\[ \Psi_{III} = Ee^{i(\omega t - k_3 x)} + Fe^{i(\omega t + k_3 x)} \]

- **Note:** (i) \( \omega \) is the same in all regions: otherwise there would be discontinuities at boundaries such that neighboring molecules would move with different frequencies. This would be unphysical.

**Example for a rightward - travelling wave on a string of 3 different densities.**
But also note: (ii) wave velocity \( v = \sqrt{\frac{\tau}{\rho}} \), so having different densities \( \rho \) in different regions implies different velocities in different regions. Then: since wave number \( k \equiv \frac{\omega}{v} \), different regions have different \( k \)'s.

For this example, we choose \( \rho_3 = \rho_1 \), so \( k_3 = k_1 \).
Procedure, continued

2) Identify boundary conditions:

\[ \begin{align*}
(1) \quad & \Psi_I = \Psi''_I \\
(2) \quad & \Psi''_I = \Psi''''_I
\end{align*} \]

\[ \Rightarrow \begin{cases}
(\Psi \text{ is an observable (an amplitude deflection) so it must be continuous.}) \\
\text{motivation below}
\end{cases} \]

\[ \begin{cases}
(3) \quad & \left. \frac{\partial \Psi}{\partial x} \right|_I = \left. \frac{\partial \Psi}{\partial x} \right|_{II} \\
(4) \quad & \left. \frac{\partial \Psi}{\partial x} \right|_{II} = \left. \frac{\partial \Psi}{\partial x} \right|_{III}
\end{cases} \]

(5) There is no boundary to the right of Region III.

Motivation for continuity of the first derivatives:

A discontinuous \( \frac{\partial \Psi}{\partial x} \) would imply an infinite \( \frac{\partial^2 \Psi}{\partial x^2} \). Through the wave equation,

\[ \frac{\partial^2 \Psi}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2} = 0 \]

infinite \( \frac{\partial^2 \Psi}{\partial x^2} \) implies infinite \( \frac{\partial^2 \Psi}{\partial t^2} \). That would be infinite acceleration which is unphysical.
Procedure, continued

3) Apply boundary conditions

Example, continued

1) \( A e^{i\omega t} + B e^{i\omega t} = C e^{i\omega t} + D e^{i\omega t} \)
\[ A + B = C + D \]

-the \( e^{i\omega t} \) factors will cancel similarly in the application of all the other boundary conditions too, so we ignore them subsequently-

2) \( C e^{-ik_2 L} + D e^{ik_2 L} = E e^{-ik_1 L} + F e^{ik_1 L} \)

3) \(- k_1 A + k_1 B = -k_2 C + k_2 D\)

4) \(- k_2 C e^{-ik_2 L} + k_2 D e^{ik_2 L} = -k_1 E e^{-ik_1 L} + k_1 F e^{ik_1 L} \)

5) No motivation for a leftware-travelling wave in Region III, so \( F = 0 \)

Notice that at this point there are 5 equations [(1) - (5)] and 6 unknowns (A-F). This is reasonable because nothing in the problem specified the amplitude A of the initial incoming wave. All other amplitudes follow from it.
Example, continued

4) Solve the 5 equations simultaneously to obtain $\frac{B}{A}$.

Then compute $\left|\frac{B}{A}\right|^2 \equiv R$, the 'reflection coefficient'
at the Region I - Region II interface.

Also find $\frac{C}{A}$ to get

$\left|\frac{C}{A}\right|^2 \equiv T$, the 'transmission coefficient' at the same interface.

We could continue similarly to find these coefficients at
the interface of Regions II and II.

At ANY interface, $T + R = 1$:

no energy is permanently trapped in the interface.
III. Phase velocity

Recall that a wave is a travelling pattern or disturbance. Information about the direction and wavelength are encoded in the phase. Let us call the phase

\[ \xi \equiv x \pm vt. \]

So for example a harmonic wave is given by

\[ \Psi = A \cos \left( \frac{\omega}{v} \xi \right) = A \cos \left( \frac{\omega}{v} x \pm \omega t \right) = A \cos (kx \pm \omega t) \]

For a fixed \( \xi \), \( \Psi \) has a fixed value (i.e., amplitude).

Our goal: to find the velocity associated with maintaining a fixed phase \( \xi \).

Consider time interval \( dt \).

What is the length increment \( dx \) needed to keep \( \xi \) constant if \( \xi \equiv x - vt \) ?

To answer this, demand that \( d\xi \equiv dx - vdt = 0 \).

Then \( \frac{dx}{dt} = v \).
For \( \frac{dx}{dt} = v \), the value of \( \Psi \) at \((x + dx, t + dt)\) will be the same as it is at \((x, t)\). The pattern moves with velocity \( v \), called the 'phase velocity.' This is the same 'v' as in

(1) the wave equation: \( \frac{\partial^2 \Psi}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2} = 0 \)

(2) \( k \equiv \frac{\omega}{v} \)

But the definition of \( v_{\text{phase}} \) is the requirement to keep \( \xi \) constant.

The statement \( v_{\text{phase}} = \frac{\omega}{k} \) is not the definition; it is a consequence.

Usually these requirements are the same, but sometimes 'constant phase' must mean \( v_{\text{phase}} = \text{Re} \left( \frac{\omega}{k} \right) \).
IV. Dispersion

When wave velocity is a function of $k$, the medium is dispersive and the wave exhibits dispersion: different frequency waves travel with different velocities. So if they all start out with crests bunched together in a packet, their crests eventually separate spatially so the wave disperses. Dispersiveness occurs because every material has its unique molecular structure (masses $m$, spring constants $K$) so unique normal modes and boundary conditions.

The loaded string is a 1-dimensional model for any material with its unique m's and K's. We now show how the modes of a loaded string lead to the condition in which $v_{\text{phase}}$ is a function of wave number $k$ for waves on the string (or, in any medium).
Let \# masses = \( n \)
separation between masses = \( d \)
mass value = \( m \)
string length = \( L = (n+1)d \)
"spring" constants = \( K \)

Recall:
The normal frequencies are [Slide 314]

\[
\omega_N = 2\sqrt{\frac{K}{m}} \sin \left( \frac{N\pi}{2n+2} \right) \quad N = 1, 2, 3, ...
\]

If the string has physical length \( L \) and is fixed at both ends, any wave on it must have nodes at least at \( x = 0 \) and \( x = L \).
So the boundary conditions are \( \Psi(0) = \Psi(L) = 0 \).

The length \( L \) must fit \( \frac{N}{2} \) wavelengths \( (N = 1, 2, 3, ...) \).

i.e., \( L = \frac{N}{2} \lambda \) for wavelength \( \lambda \)
But wavelength \( \lambda = \frac{2\pi}{k} \) where \( k \) is wave number

and \( L = (n + 1)d \)

So \( (n + 1)d = L = \frac{N}{2} \cdot \frac{2\pi}{k} \)

\[
\frac{kd}{2} = \frac{N\pi}{2(n + 1)}
\]

Then \( \omega = 2\sqrt{\frac{K}{m}} \sin\left(\frac{kd}{2}\right) \)

Then \( v_{\text{phase}} = \frac{\omega}{k} = \frac{2}{k} \sqrt{\frac{K}{m}} \sin\left(\frac{kd}{2}\right) \), a function of \( k \).