I. Energy conservation and Bernoulli's Equation
II. Bulk modulus and the speed of sound
III. Viscosity
I. Energy conservation and Bernoulli's Equation

Consider a tube of moving fluid with constant density $\rho$

By the Continuity Equation, we know that mass is conserved. Consider the time interval $dt$.

At End 1, the fluid travels distance $v_1 dt$.
At End 2, the fluid travels distance $v_2 dt$. 
The total amount of mass $dm$ that enters and exits during interval $dt$ is

$$dm = \frac{mass}{volume} \times area \times \frac{length}{time} \times time$$

$$= \rho \cdot A_1 \cdot v_1 \cdot dt \quad \text{(enters)}$$

$$= \rho \cdot A_2 \cdot v_2 \cdot dt \quad \text{(exits)}$$

During the flow from End 1 to End 2, the fluid may:
- change kinetic energy ($KE_1 \Rightarrow KE_2$)
- change potential energy ($U_1 \Rightarrow U_2$)
- have work $W_1$ done on it by pressure $P_1$
- have work $W_2$ done on it by pressure $P_2$

(Note: work done on the fluid is positive; work done by the fluid is negative.)
By conservation of energy, we expect

\[ KE_1 + U_1 + W_1 = KE_2 + U_2 + W_2 \]  

'Eq 1'

\[ KE_1 = (dm) \frac{v_1^2}{2} \]

\[ KE_2 = (dm) \frac{v_2^2}{2} \]

\[ U_1 = (dm) gh_1 \]

\[ U_2 = (dm) gh_2 \]

\[ W_1 = \text{Force} \cdot \text{distance} \]

Note pressure = \( \frac{\text{force}}{\text{area}} \)

and velocity = \( \frac{\text{distance}}{\text{time}} \)

So \( W_1 = P_1 A \cdot v_1 dt \)
\[ W_1 = P_1 A \cdot v_1 dt \]
\[ = P_1 A \cdot v_1 dt \cdot \frac{\rho}{\rho} \]
\[ = \frac{P_1 (dm)}{\rho} \]
Similarly,
\[ W_2 = \frac{P_2 (dm)}{\rho} \]
Substitute all of these into Eq 1:
\[ (dm) \frac{v_1^2}{2} + (dm) gh_1 + \frac{P_1 (dm)}{\rho} = (dm) \frac{v_2^2}{2} + (dm) gh_2 + \frac{P_2 (dm)}{\rho} \]
\[ \frac{v_1^2}{2} + gh_1 + \frac{P_1}{\rho} = \frac{v_2^2}{2} + gh_2 + \frac{P_2}{\rho} \]
That is, at any point along the path:
\[ \frac{v^2}{2} + gh + \frac{P}{\rho} = \text{constant} \]
Rewriting: \[ \frac{v^2}{2} + gh + \frac{P}{\rho} = \text{constant} \]

Symon separates this into 2 terms: \( \frac{p}{\rho} + u \)

Symon calls this \(-G\)

Bernoulli's Equation: as velocity increases, pressure decreases.
II. Bulk modulus and the speed of sound

Recall Euler's equation of motion for a moving fluid [Slide 418]

\[
\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} + \frac{\nabla p}{\rho} = \frac{\vec{f}}{\rho}
\]

\[F_{ext}/Vol\]

pressure
density

Let the fluid be at rest and not accelerating:

\[
\frac{\nabla p_0}{\rho_0} = \frac{\vec{f}_0}{\rho_0}
\]

'Eq 0', Euler's Equation for a fluid at rest

Now induce some sound waves: perturb the fluid slightly so that

\[p_0 \Rightarrow p_0 + p' \equiv p\]
\[\rho_0 \Rightarrow \rho_0 + \rho' \equiv \rho\]

Note these prime symbols do NOT indicate a derivative. These are just Symon's notation for "small perturbation"

\[p' \ll p_0\]
\[\rho' \ll \rho_0\]
For small velocity and small perturbations, we will assume

\[(p')^2 = 0\]

\[(\rho')^2 = 0\]

\[p'v = 0\]

\[v \cdot \nabla v = 0\]

Substitute these into Euler's Equation for moving fluids

\[\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} + \frac{1}{(\rho_0 + \rho')} \cdot \nabla (p_0 + p') = \frac{1}{(\rho_0 + \rho')} \cdot \vec{f}\]

\[\{ \approx 0, \text{neglect} \}\]

Expand these as

\[\frac{1}{(\rho_0 + \rho')} = \frac{1}{\rho_0 \left(1 + \frac{\rho'}{\rho_0}\right)} = \frac{1}{\rho_0} \left(1 - \frac{\rho'}{\rho_0} + \ldots\right)\]
All that remains is:

\[
\frac{\partial \vec{v}}{\partial t} \approx -\frac{1}{\rho_0} \nabla p'
\]

'Eq 1'

Now make the same substitutions and approximations in the Continuity Equation [Slide 416],

\[
\frac{\partial \rho}{\partial t} + \vec{V} \cdot (\rho \vec{v}) = 0
\]

\[
\frac{\partial (\rho_0 + \rho')}{\partial t} + \vec{V} \cdot (\rho_0 + \rho') \vec{v} = 0
\]
\[
\frac{\partial \rho_0}{\partial t} + \frac{\partial \rho'}{\partial t} + \vec{\nabla} \cdot (\rho_0 \vec{v}) + \vec{\nabla} \cdot (\rho' \vec{v}) = 0
\]

\[
\begin{align*}
\{ & \text{at rest, } \\
\{ \rho_0 \neq \rho_0(t) \} & \{ \approx 0, \text{ neglect} \}
\end{align*}
\]

We are left with:

\[
\frac{\partial \rho'}{\partial t} + \vec{\nabla} \cdot (\rho_0 \vec{v}) = 0
\]

\[
\frac{\partial \rho'}{\partial t} + \rho_0 \vec{\nabla} \cdot \vec{v} + \vec{v} \cdot \vec{\nabla} \rho_0 = 0
\]

\[
\begin{align*}
\{ & \text{at rest, } \\
\{ \rho_0 \neq \rho_0(x_i) \}
\end{align*}
\]

We are finally left with:

\[
\frac{\partial \rho'}{\partial t} = -\rho_0 \vec{\nabla} \cdot \vec{v} \quad \quad \text{'Eq 2'}
\]
Recall [Slide 398]:

\[ \text{strain} = -\frac{1}{B} dp = \frac{dV}{V} = -\frac{d\rho}{\rho} \]

In this section, \( dp \) is \( p' \) (means "small pressure") and \( d\rho \) is \( \rho' \) (means "small density") so removing the unnecessary \( \frac{dV}{V} \)

and recalling that \( B \) is bulk modulus,

\[ \text{strain} = -\frac{1}{B} p' = -\frac{\rho'}{\rho_0} \]

That is:

\[ \rho' = \frac{\rho_0 p'}{B} \]

Substitute this into Eq 2:

\[ \frac{\partial}{\partial t}\left( \frac{\rho_0 p'}{B} \right) = -\rho_0 \nabla \cdot \vec{v} \]
\[ \frac{\partial p'}{\partial t} = -B \nabla \cdot \vec{v} \]  

\(\text{'Eq 3'}\)

Rewrite Eq 1:
\[ \frac{\partial \vec{v}}{\partial t} \approx - \frac{1}{\rho_0} \nabla p' \]

Equations 1 and 3 are the fundamental equations for sound waves, coupling pressure and velocity variations in space and time.

Next goal: Find an expression for the speed of sound. Combine Eq 1 and Eq 3 to get an equation of the form
\[ \nabla^2 (\text{something}) - \frac{1}{(\text{something else})^2} \frac{\partial^2 (\text{something})}{\partial t^2} = 0 \]

Since this is the wave equation, the "something else" must be the speed of sound.
To uncouple the equations, take $\vec{\nabla} \cdot (\text{Eq 1})$:

$$\vec{\nabla} \cdot \left( \frac{d\vec{v}}{dt} \right) = \vec{\nabla} \cdot \left( -\frac{1}{\rho_0} \vec{\nabla} p' \right)$$

$$\frac{d}{dt} \left( \vec{\nabla} \cdot \vec{v} \right) = -\frac{1}{\rho_0} \nabla^2 p' \quad \text{'Eq 4'}$$

Now take $\frac{d}{dt}$ (Eq 3):

$$\frac{d}{dt} \left( \frac{dp'}{dt} \right) = \frac{d}{dt} \left( -B \vec{\nabla} \cdot \vec{v} \right)$$

$$\frac{d^2 p'}{dt^2} = -B \frac{d}{dt} \left( \vec{\nabla} \cdot \vec{v} \right)$$

Plug in Eq 4 here

$$\frac{d^2 p'}{dt^2} = -B \left( -\frac{1}{\rho_0} \nabla^2 p' \right)$$

$$\frac{d^2 p'}{dt^2} = \frac{B}{\rho_0} \nabla^2 p'$$
\[
\frac{d^2 p'}{dt^2} = \frac{B}{\rho_0} \nabla^2 p'
\]

\[
\nabla^2 p' = \frac{1}{B} \frac{d^2 p'}{dt^2} \frac{1}{\rho_0}
\]

We see that

\[(\text{speed of sound})^2 = \frac{B}{\rho_0}\]

speed of sound = "c" = \[\sqrt{\frac{B}{\rho_0}}\]

**Vocabulary:**

Mach number = \[\frac{\nu}{c}\], where \(\nu\) is whatever velocity is characteristic of the problem (for example, average fluid velocity). Typically for \(\nu < c\), the fluid can be treated as incompressible.
What to use for bulk modulus $B$?

Recall $B \equiv -\frac{dp}{dV} V$

Recall from Thermodynamics that for physical many-particle systems, most transformations are isothermal (constant temperature) or adiabatic (no heat transfer). These two have different relationships between $p$ and $V$, so their $\frac{dp}{dV}$'s are different.
**Isothermal**
Constant temperature, so the ideal gas law $pV = RT$ becomes: $pV = \text{constant}$.

**Adiabatic**
No heat transfer $pV^\gamma = \text{constant}$ where $\gamma = c_p / c_v$

These are specific heats at constant volume or constant pressure.

---To find $B$, differentiate both sides---

\[ d(pV) = d(\text{constant}) \]
\[ dpV + pdV = 0 \]
\[ -\frac{dp}{dV} V = p \]

---Conclude:---

$B_{\text{isothermal}} = p$

$B_{\text{adiabatic}} = p\gamma$
Empirical data on the speed of sound for different materials has shown that in most cases $B_{\text{sound, transmission}} = B_{\text{adiabatic}}$ so if in doubt, use $B = p\gamma$. 
III. Viscosity
...the internal friction that inhibits the flow of a fluid.
To characterize it, consider 2 parallel glass plates of area \( A \) separated by a distance \( d \). A fluid is between them.

Bottom plate has velocity \( = 0 \).
Top plate has velocity \( \bar{v} = v\hat{x} \).
To maintain this there must be a force \( F_v \) applied across the top plate.
That is, it must be shearing.

Fluid directly below top plate has vel \( \approx v \).
Fluid directly above bottom plate has vel \( \approx 0 \).
We expect a smooth increase of fluid velocity magnitude from bottom to top.
We expect the force $F_x$ needed to produce this velocity pattern to be

$$F_x \propto A$$

$$F_x \propto v$$

$$F_x \propto \frac{1}{d}$$

Put these together:

$$F_x = \eta A \frac{v}{d}$$

$\eta$, called the coefficient of viscosity, is a constant of proportionality that depends upon the fluid.

Note that "poise," the units of $\eta$ are not MKS: \( \frac{\text{kg}}{\text{m} \cdot \text{s}} \) = 10 poise.

Most materials have coefficients in the range of

centipoise = $10^{-2}$ poise = $10^{-3}$ kg/m-s
Approach the limit where \( d \Rightarrow \delta y \)
and \( v \Rightarrow \delta v_x \)

Then \( F_x = \eta A \frac{\delta v_x}{\delta y} \)

\[ \eta = \frac{F_x / A}{\delta v_x / \delta y} \]

Typical values: water = 1 centipoise
blood = 4 centipoise
ketchup = \( 5 \times 10^4 \) centipoise
peanut butter = \( 1.5 \times 10^5 \) centipoise
road tar = \( 10^{10} \) centipoise
I. Poiseuille's Law

II. Sound waves
I. Poiseuille's Law

Goal: show that \textbf{for viscous fluid, the throughput through a pipe is proportional to the (pipe radius)}^{4}.

Consider a pipe of radius $a$, length $\ell$.

Assume flow is laminar: in sheets, layers

\begin{equation}
Q = \frac{\text{volume expelled}}{\text{time}}
\end{equation}

Define $Q = \text{throughput of the pipe} = \text{discharge rate} = \frac{\text{volume expelled}}{\text{time}}$

\begin{equation}
Q = \nu A
\end{equation}

- cross-sectional area of pipe
- fluid velocity, which depends on radius

So to find throughput $Q$, we need to find $\nu$.

The issue is: in a viscous fluid, $\nu$ depends on location in the pipe.
Consider the fluid in a layer from $r = 0$ to $r = r'$. The frictional force on its boundary is

$$F_{\text{visc}} = \eta A \frac{dv}{dr'}$$

$$= \eta (2\pi r' \ell) \frac{dv}{dr'}$$

If the flow is at constant speed, acceleration = 0, so

$$F_{\text{tot}} = ma = 0.$$ 

In that case $F_{\text{visc}}$ must be balanced by another force: this is associated with the pressure differential between the 2 ends of the pipe.
The pressure difference along the pipe is \( \Delta p = \frac{F}{A} = \frac{F_{\text{pressure}}}{\pi a^2} \).

\[
F_{\text{visc}} + F_{\text{pressure}} = 0 :
\]

\[
\eta 2\pi r' \ell \frac{dv}{dr'} + \Delta p \pi a^2 = 0
\]

\[
dv = -\frac{1}{2\eta} \frac{\Delta p}{\ell} r dr
\]

To find the velocity of the fluid at radius \( r' \), integrate from \( r' \) to pipe boundary \( a \), then plug in \( v(a) = 0 \):

\[
\int_{r'}^{a} dv = -\frac{1}{2\eta} \frac{\Delta p}{\ell} \int_{r'}^{a} r dr
\]

\[
v(a) - v(r') = -\frac{1}{2\eta} \frac{\Delta p}{\ell} \frac{a^2 - r'^2}{2}
\]

\[
v(r') = \frac{\Delta p}{4\eta \ell} \left( a^2 - r'^2 \right)
\]
Recall $Q = vA$

Let $dQ$ be throughput from a ring at radius $r$ of thickness $dr$.

$$dQ = v dA$$

$$= \frac{\Delta p}{4 \eta \ell} \left( a^2 - r^2 \right) (2\pi r dr)$$

Total throughput:

$$Q = \int_0^a dQ = \frac{\Delta p}{4 \eta \ell} \cdot 2\pi \int_0^a (a^2 - r^2) r \, dr$$

$$= \frac{\pi a^4 \Delta p}{8 \eta \ell} \quad \text{Poiseuille's Law, valid only for laminar flow.}$$
Define: Reynolds number \( R \), for fluid velocity \( v \), density \( \rho \), viscosity \( \eta \), flowing in a pipe of diameter \( 2a \):

\[
R \equiv \frac{\rho v}{\eta}(2a)
\]

\( R \) is dimensionless.

\( R \leq 2000 \): laminar flow

\( R > 2000 \): turbulent flow
II. Sound waves

Recall that the 2 fundamental equations for sound are

\[ \frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho_0} \vec{\nabla} p' \]  

'Eq 1'

small pressure fluctuations from ambient fluid density in equilibrium fluid velocity (not wave velocity)

and

\[ \frac{\partial p'}{\partial t} = -B \vec{\nabla} \cdot \vec{v} \]  

'Eq 2'

We eliminated \( \vec{v} \) from them to get

\[ \nabla^2 p' - \frac{1}{B} \frac{\partial^2 p'}{\partial t^2} = 0 \]  

'Eq 3'
We could instead have eliminated $p'$ from them to get

$$\nabla^2 v - \frac{1}{B} \frac{\partial^2 v}{\partial t^2} = 0 \quad \text{'Eq 4'}$$

Goal now:

1) show that the solutions are plane waves in both $p'$ and $v$
2) show that $\tilde{v}$ is parallel to the direction of wave propagation, that is: sound waves are longitudinal, not transverse
Let us understand plane waves in 2 steps:
first describe the plane, then describe the wave.
Consider a plane moving in the \( \hat{n} \) direction:

Identify 2 points on it,
\[ \vec{r} = (x, y, z) \] and
\[ \vec{r}_0 = (x_0, y_0, z_0) \]
They are joined by vector \( (\vec{r} - \vec{r}_0) \)
By definition of a plane (i.e., flat surface), for any \( \vec{r} \) and \( \vec{r}_0 \) on the plane, 
\((\vec{r} - \vec{r}_0)\) must be \( \perp \) to \( \hat{n} \).

So \((\vec{r} - \vec{r}_0) \cdot \hat{n} = 0\)

\[
(x - x_0)n_x + (y - y_0)n_y + (z - z_0)n_z = 0
\]

\[
xn_x + yn_y + zn_z = x_0n_x + y_0n_y + z_0n_z
\]

This is a constant that defines the orientation of the plane.

So a plane satisfies the equation:

\( \hat{n} \cdot \vec{r} = \text{constant, in space, at any time} \)

That was the plane. Now the wave:

Recall that any function that maintains a constant phase as it travels is, by definition, a wave. Let the phase be

\( \xi \equiv \hat{n} \cdot \vec{r} - ct \)
One way to describe sound is as a pressure wave, which satisfies Eq 3 on Slide 450. The solution to that should look like

\[ p' = f(\hat{n} \cdot \vec{r} - ct) \]

- \( f \) is an arbitrary shape
- \( \hat{n} \cdot \vec{r} \) builds in the nature of a plane

As with 1-dimensional waves on strings, the spatial (\( \vec{r} \)) and temporal (\( t \)) parts of the phase have to be coupled in order for it to propagate. We name the coupling parameter "c."

We saw on Slide 372 that this coupling is the phase velocity, which is the velocity that a viewer traveling alongside the wave must have in order to see the phase remain constant.
We next demonstrate directly that this \( p' = f(\hat{n} \cdot \vec{r} - ct) = f(\xi) \) satisfies the wave equation (Eq 3 on Slide 450): \( \nabla^2 p' - \frac{1}{B} \frac{\partial^2 p'}{\partial t^2} = 0 \).

Compute the LHS:

If \( p' = f(\xi) \),

\[
\nabla p' = \frac{df}{d\xi} \nabla \xi = \frac{df}{d\xi} \hat{n}
\]

Then \( \nabla^2 p' = \nabla \cdot \left( \frac{df}{d\xi} \hat{n} \right) = \frac{d}{d\xi} \left( \frac{df}{d\xi} \hat{n} \right) \cdot \nabla \xi
\]

\[
= \begin{bmatrix}
\frac{d^2 f}{d\xi^2} \hat{n} + \frac{df}{d\xi} \frac{d\hat{n}}{d\xi} \\
0 & \hat{n}
\end{bmatrix} \cdot \nabla \xi
\]

\[
= \frac{d^2 f}{d\xi^2} \hat{n} \cdot \hat{n} = \frac{d^2 f}{d\xi^2}
\]
Similarly compute the RHS,

\[ \frac{\partial^2 p'}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial \xi^2} \]

We conclude that the function \( p' = f(\hat{n} \cdot \vec{r} - ct) \) satisfies the wave equation \( \nabla^2 p' - \frac{1}{\left( \frac{B}{\rho_0} \right)} \frac{\partial^2 p'}{\partial t^2} = 0 \) for any \( f \), if \( c = \sqrt{\frac{B}{\rho_0}} \).
By a similar treatment we can solve Eq 4,

\[
\nabla^2 \tilde{v} - \frac{1}{B} \frac{\partial^2 \tilde{v}}{\partial t^2} = 0
\]

with any wave that has the form \( \tilde{v} = \tilde{h}(\hat{n}' \cdot \vec{r} - ct) \)

This is phase \( \xi' \).

Note that the direction of this plane's motion is \( \hat{n}' \), not a priori the same as \( \hat{n} \).

Note this wave \( \tilde{h} \) is a vector function, conceptually like \( f \) but with 3 equations implied: one each for \( v_x, v_y, \) and \( v_z \).

We have met Goal 1 on Slide 451, which was to show that the solutions to the sound wave equation in 3-d are plane waves in \( p' \) and \( v \).

Now we go after Goal 2, to show that these waves are longitudinal.
Recall Eq 1 [Slide 450]
\[ \frac{\partial \tilde{v}}{\partial t} = \frac{1}{\rho_0} \tilde{\nabla} p' \] 'Eq 1'

Plug in \( p' = f(\xi) \), \( \tilde{v} = \tilde{h}(\xi') \), and \( c = \sqrt{\frac{B}{\rho_0}} \):

\[ \frac{\partial \tilde{v}}{\partial t} = \frac{\partial \tilde{h}}{\partial \xi'} \frac{\partial \xi'}{\partial t} = \frac{\partial \tilde{h}}{\partial \xi'} (-c) = - \frac{\partial \tilde{h}}{\partial \xi'} \sqrt{\frac{B}{\rho_0}} \]

\[ \nabla p' = \frac{\partial f}{\partial \xi} \nabla \xi = \frac{\partial f}{\partial \xi} \hat{n} \]

Eq 1 becomes:

\[ - \frac{\partial \tilde{h}}{\partial \xi'} \sqrt{\frac{B}{\rho_0}} = - \frac{1}{\rho_0} \frac{\partial f}{\partial \xi} \hat{n} \]

\[ \frac{\partial \tilde{h}}{\partial \xi'} = \frac{\hat{n}}{\sqrt{B \rho_0}} \frac{\partial f}{\partial \xi} \]

This must be true for all \( \tilde{r} \), at all \( t \)
\[
\frac{\partial \tilde{h}}{\partial \xi'} = \frac{\hat{n}}{\sqrt{B\rho_0}} \frac{\partial f}{\partial \xi}
\]

'Eq 5'

The RHS is a function of \(\xi\) and is constant, for constant \(\xi\).
So the LHS must be constant when \(\xi\) is constant.
So \(\xi'\) must be a function of \(\xi\), and the range of allowed functions
is limited by maintaining the plane wave.
For simplicity let us choose \(\xi' = \xi\).
Then \(\hat{n}' \cdot \vec{r} - ct = \hat{n} \cdot \vec{r} - ct\)
so \(\hat{n}' = \hat{n}\)

Now rewrite Eq 5, setting \(\xi' = \xi\).
\[
\frac{\partial \tilde{h}}{\partial \xi} = \frac{\hat{n}}{\sqrt{B\rho_0}} \frac{\partial f}{\partial \xi}
\]
Integrate, set integration constant = 0 because \(p' = 0\) when \(v = 0\)
(that is, when there is no wave).
\[ \vec{h} = \frac{\hat{n}}{\sqrt{B\rho_0}} f \]

But \( \vec{h} = \vec{v} \) and \( f = p' \):

\[ \vec{v} = \frac{\hat{n}}{\sqrt{B\rho_0}} p' \]

Conclusions:

By construction, \( \hat{n} \) is the direction of the wave's velocity, which is

\[ c\hat{n} = \sqrt{\frac{B}{\rho_0}} \hat{n} \]

Here we see that \( \hat{n} \) is also the direction of the fluid particles' velocity \( \vec{v} \). This demonstrates Goal 2: sound waves are longitudinal; and the pressure variations \( p' \) arise from the velocity variations.
I. Wave vector
II. Sound wave power
III. Other wave forms
IV. Normal modes of a fluid in a box
V. Features of sound waves in cavities
I. Wave vector

Define the wave vector $\vec{k}$ for 3-dimensional waves (like sound) analogously to the wave number $k$ (for 1-dimensional waves on string):

$$\vec{k} \equiv \frac{\omega}{c} \hat{n}$$

$\hat{n}$ is the direction of propagation of the wave front

$\omega$ is the wave's angular frequency

$c$ here is the speed of the wave, not the speed of light

Then $p' = f (\hat{n} \cdot \vec{r} - ct)$ becomes

$$= f \left[ \frac{c}{\omega} \vec{k} \cdot \vec{r} - ct \right]$$

$$= f \left[ \frac{c}{\omega} (\vec{k} \cdot \vec{r} - \omega t) \right]$$

But $c$ and $\omega$ are fixed.

$$= f (\vec{k} \cdot \vec{r} - \omega t)$$
II. Sound wave power

The force that causes the pressure variations moves the particles back and forth.

\[ \text{Power} = \vec{F} \cdot \vec{v} \]

\( \vec{F} \parallel \vec{v} \), so power is transmitted.

Symon uses the confusing notation \( P \) to mean "Power per area"

So \( P = \frac{\text{Power}}{\text{Area}} = \frac{\vec{F}}{\text{Area}} \cdot \vec{v} \)

\[ = \frac{F}{\text{Area}} v \quad \text{(scalars)} \]

\[ = p \cdot v \quad \text{for pressure} \ p \]

Average over one cycle of the wave:

\[ \langle P \rangle = \langle (p_0 + p')v \rangle \]

But \( p_0 \) is equilibrium pressure, a constant, and \( \langle v \rangle = 0 \)
\[ \langle P \rangle = \langle p' \nu \rangle \]

But \( \bar{\nu} = \frac{p'}{\sqrt{B \rho_0}} \hat{n} \)

\[ \langle P \rangle = \left\langle \frac{(p')^2}{\sqrt{B \rho_0}} \right\rangle \]

\( \sqrt{B \rho_0} \) is a constant that can come outside the averaging operation

\[ \langle P \rangle = \frac{1}{\sqrt{B \rho_0}} \langle (p')^2 \rangle \]

The average power per area of a sound wave is the energy per unit area per second, travelling in direction \( \hat{n} \).
III. Other wave forms

Wave shapes that satisfy the wave equation

\[ \nabla^2 p' - \frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2} = 0 \]

take their form from the shape of the initial disturbance. Thus they could be:

- spherical: \( p' = \frac{1}{r} f(r - ct) \)

- cylindrical: \( p' = \frac{1}{\sqrt{r}} f(r - ct) \)

What motivates the radial factor in front of function \( f \)?

In a classical wave, energy = \(|amplitude|^2\).

As spherical or cylindrical waves move outward, they cover an increasingly larger area, so their amplitude per area must decrease in order to guarantee energy conservation.

The area of a wave front in a spherical system grows as \( r^2 \).

The area of a wave front in a cylindrical system grows as \( r \).
IV. Normal modes of a fluid in a box

Relevance: acoustics and solids, but also electromagnetic waves in waveguides and quantum mechanical particles' probability density in a potential well.

Consider a fluid in a box of volume \( L_x \cdot L_y \cdot L_z \).

The fluid can oscillate but must maintain the boundary conditions.

Oscillations described by the "sound wave" equation apply to any 3-dimensional wave:

\[
\nabla^2 p' - \frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2} = 0
\]

Our goal: solve this by separation of variables (in 2 steps), for various boundary conditions.

\[
\nabla^2 p' = \frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2}
\] 'Eq 1'

Step 1: separate spatial dependence from time dependence.

Guess \( p' = \psi(x,y,z)T(t) \)
\( p' = \psi(x, y, z) T(t) \)

\[ \begin{aligned}
\nabla^2 p' &= \nabla^2 \psi \cdot T \\
\frac{\partial^2 p'}{\partial t^2} &= \psi \frac{\partial^2 T}{\partial t^2}
\end{aligned} \]

Substitute these into Eq 1.

\[ \begin{aligned}
\nabla^2 \psi \cdot T &= \frac{1}{c^2} \psi \frac{\partial^2 T}{\partial t^2} \\
\frac{\partial^2 T}{\partial t^2} &= \frac{\omega^2}{c^2} \psi
\end{aligned} \]

\[ \div \text{ by } \psi T: \]

\[ \begin{aligned}
\nabla^2 \psi &= \frac{1}{c^2 T} \frac{\partial^2 T}{\partial t^2} \\
\frac{\partial^2 T}{\partial t^2} &= \frac{\omega^2}{c^2}
\end{aligned} \]

Both sides must equal the same constant. Call it "\(- \frac{\omega^2}{c^2}\)"

**LHS**

\[ \begin{aligned}
\nabla^2 \psi &= - \frac{\omega^2}{c^2} \\
\nabla^2 \psi + \frac{\omega^2}{c^2} \psi &= 0 \quad \text{'Eq 2'}
\end{aligned} \]

**RHS**

\[ \begin{aligned}
\frac{1}{c^2 T} \frac{\partial^2 T}{\partial t^2} &= - \frac{\omega^2}{c^2} \\
\frac{\partial^2 T}{\partial t^2} + \omega^2 T &= 0
\end{aligned} \]

Solving Eq 2 requires Step 2.

\[ T = A \cos \omega t + B \sin \omega t \]
Step 2: separate all the spatial dimensions

Guess $\psi = X(x) \cdot Y(y) \cdot Z(z)$

So $\nabla^2 \psi = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) XYZ$

$$= \left( \frac{\partial^2 X}{\partial x^2} \right) YZ + X \left( \frac{\partial^2 Y}{\partial y^2} \right) Z + XY \left( \frac{\partial^2 Z}{\partial z^2} \right)$$

Eq 2 becomes:

$$\left( \frac{\partial^2 X}{\partial x^2} \right) YZ + X \left( \frac{\partial^2 Y}{\partial y^2} \right) Z + XY \left( \frac{\partial^2 Z}{\partial z^2} \right) + \frac{\omega^2}{c^2} \cdot XYZ = 0$$

$\div$ by $XYZ$

$$\frac{1}{X} \left( \frac{\partial^2 X}{\partial x^2} \right) + \frac{1}{Y} \left( \frac{\partial^2 Y}{\partial y^2} \right) + \frac{1}{Z} \left( \frac{\partial^2 Z}{\partial z^2} \right) = -\frac{\omega^2}{c^2}$$

\[ \begin{align*}
\text{Term 1} & \quad \text{Term 2} & \quad \text{Term 3} & \quad \text{To ensure that this is true for all } (x, y, z), \\
\left\{ \text{"-} k_x^2 \right\} & \quad \left\{ \text{"-} k_y^2 \right\} & \quad \left\{ \text{"-} k_z^2 \right\} & \quad \text{each term on the LHS must be a constant}
\end{align*} \]
So \(-k_x^2 - k_y^2 - k_z^2 = -\frac{\omega^2}{c^2}\)

\[k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}\] 'Eq 6'

Now the partial derivatives become totals:

Term 1: \(\frac{1}{X} \left( \frac{\partial^2 X}{\partial x^2} \right) = -k_x^2\)

\[\frac{\partial^2 X}{\partial x^2} + k_x^2 X = 0\]

\[X = C_x \cos k_x x + D_x \sin k_x x\]

Similarly,

Term 2 produces

\[Y = C_y \cos k_y y + D_y \sin k_y y\]

Term 3 produces

\[Z = C_z \cos k_z z + D_z \sin k_z z\]
Put this together to get

\[ p' = \psi T = XYZT \]

We now consider the boundary conditions for a *closed* rectangular box: Particles' velocity cannot transport them beyond the walls:

1) \[ v_x (x = 0) = 0 \]
2) \[ v_x (x = L_x) = 0 \]
3) \[ v_y (y = 0) = 0 \]
4) \[ v_y (y = L_y) = 0 \]
5) \[ v_z (z = 0) = 0 \]
6) \[ v_z (z = L_z) = 0 \]
To prepare to apply boundary condition (1), recall one of the 2 fundamental equations for sound waves [Eq 1 on Slide 432]:

\[ \frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho_0} \nabla p' \]

In 1-dimension:

\[ \frac{\partial v_x}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x} \]

Substitute \( p' = XYZT \).

Note:

\[ \frac{\partial p'}{\partial x} = k_x \cdot \left[ -C_x \sin k_x x + D_x \cos k_x x \right] \cdot YZT \]

Integrate, choose constant of integration = 0:

\[ v_x = -\frac{k_x}{\rho_0} \left[ -C_x \sin k_x x + D_x \cos k_x x \right] \cdot YZ \cdot \int dt \left[ A \cos \omega t + B \sin \omega t \right] \]

\[ \frac{1}{\omega} A \sin \omega t - \frac{1}{\omega} B \cos \omega t \]
\[ v_x = -\frac{k_x Y Z}{\rho_0 \omega}[ -C_x \sin k_x x + D_x \cos k_x x ][ A \sin \omega t - B \cos \omega t ] \]

Now apply boundary condition (1). To achieve it, \( D_x = 0 \).

Then \( X = C_x \cos k_x x \).

Notice this means that function \( X \) is maximum at \( x=0 \): an "anti-node"

To emphasize: pressure \( p' \propto X \) is a maximum at \( x=0 \).

Thus, the statement "\( |p(x=0)| = \text{max} \)" is an alternative form of Boundary Condition (1).

Similarly, we can replace Boundary Condition (2) \( [v_x(x = L_x) = 0] \) with "\( |p(x = L_x)| = \text{max} \)"

\[
\cos k_x L_x = \pm 1 \\
k_x L_x = \ell \pi \quad (\ell = 0, 1, 2, \ldots) \\
k_x = \frac{\ell \pi}{L_x}
\]
Similarly for functions $Y$ and $Z$ we find:

$$D_y = 0$$
$$D_z = 0$$

$$k_y = \frac{m\pi}{L_y} \quad (m = 0, 1, 2, ...)
$$

$$k_z = \frac{n\pi}{L_z} \quad (n = 0, 1, 2, ...)
$$

Now recall the constraint, Eq 6 [Slide 469]:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}
$$

Solve for allowed frequencies $\omega$:

$$\omega = c \left[ k_x^2 + k_y^2 + k_z^2 \right]^{\frac{1}{2}}
$$

$$= c\pi \left[ \frac{\ell^2}{L_x^2} + \frac{m^2}{L_y^2} + \frac{n^2}{L_z^2} \right]^{\frac{1}{2}} \quad \text{Call these the } \omega_{\ell mn}
$$

Every combination of integers $(\ell, m, n)$ except $(0, 0, 0)$ gives an allowed normal mode of this resonant closed rectangular cavity.
Build the pressure wavefunction from the information we have up to this point:

\[ p' = TXYZ \]

\[ = \left( A \cos \omega_{\ell mn} t + B \sin \cos \omega_{\ell mn} t \right) C_x \cos \frac{\ell \pi x}{L_x} C_y \cos \frac{m \pi y}{L_y} C_z \cos \frac{n \pi z}{L_z} \]

Rename \( AC_x C_y C_z \Rightarrow A \)

Rename \( BC_x C_y C_z \Rightarrow B \)

\[ p' = \left( A \cos \omega_{\ell mn} t + B \sin \cos \omega_{\ell mn} t \right) \cos \frac{\ell \pi x}{L_x} \cos \frac{m \pi y}{L_y} \cos \frac{n \pi z}{L_z} \]

Similarly,

\[ v_x = \frac{\ell \pi}{L_x \rho_0 \omega_{\ell mn}} \left[ A \sin \omega_{\ell mn} t - B \cos \omega_{\ell mn} t \right] \sin \frac{\ell \pi x}{L_x} \cos \frac{m \pi y}{L_y} \cos \frac{n \pi z}{L_z} \]

\[ v_y = \frac{m \pi}{L_y \rho_0 \omega_{\ell mn}} \left[ A \sin \omega_{\ell mn} t - B \cos \omega_{\ell mn} t \right] \cos \frac{\ell \pi x}{L_x} \sin \frac{m \pi y}{L_y} \cos \frac{n \pi z}{L_z} \]

\[ v_z = \frac{n \pi}{L_z \rho_0 \omega_{\ell mn}} \left[ A \sin \omega_{\ell mn} t - B \cos \omega_{\ell mn} t \right] \cos \frac{\ell \pi x}{L_x} \cos \frac{m \pi y}{L_y} \sin \frac{n \pi z}{L_z} \]

These are the four equations of motion for a fluid in a closed rectangular cavity.
II. Features of sound waves in cavities

1) If a cavity is long and narrow ("organ pipe"), then $L_x \gg L_y$ or $L_z$, and the dominant tones will occur for $m = 0$, $n = 0$, and all $\ell$.

2) If the cavity has one end open, that boundary condition is harder to define. Typically we choose $p'$ at that end to be a node. (Lord Rayleigh found that the node actually occurs $\frac{8d}{3\pi}$ beyond the end, where $d$ characterizes the diameter of a circular pipe.)

Consider a rectangular pipe with one end open:
Recall \( p' = XYZT \)

We focus here on function \( X = C_x \cos k_x x + D_x \sin k_x x \)

Boundary condition (1): \( X(x = 0) = 0 \): The open face is a pressure node.

Requires \( C_x = 0 \)

Boundary condition (2): \( X(x = L_x) = \text{max} \): The closed face is a velocity antinode, as previously

\[
\sin k_x L_x = \pm 1
\]

\[
k_x L_x = \frac{(2\ell + 1)\pi}{2}
\]

\[
k_x = \frac{(2\ell + 1)\pi}{2L_x}
\]

The boundary conditions on functions \( Y \) and \( Z \) are as previously, so

\[
\omega = c \left[ k_x^2 + k_y^2 + k_z^2 \right]^{\frac{1}{2}} \implies \omega_{\ell mn} = c\pi \left[ \frac{(2\ell + 1)^2}{4L_x^2} + \frac{m^2}{L_y^2} + \frac{n^2}{L_z^2} \right]^{\frac{1}{2}}
\]

These are the normal frequencies of an open-ended pipe.
3) For sound propagating through a pipe that is open at both ends: Choose the coordinate system such that the open ends are along $\hat{z}$.

\[ p' = XYZT \]

\[ = A \cos \left( \frac{\ell \pi x}{L_x} \right) \cos \left( \frac{m \pi y}{L_y} \right) \cos (k_z z - \omega t) \]

These are standing waves \hspace{1cm} This is the propagating term

It is still true that \[ \omega = c \left[ k_x^2 + k_y^2 + k_z^2 \right]^{1/2}, \] so

\[ k_z = \pm \sqrt{\frac{\omega^2}{c^2} - \left( \frac{\ell \pi}{L_x} \right)^2 - \left( \frac{m \pi}{L_y} \right)^2} \hspace{1cm} \ell, m = 0, 1, 2, ... \]

\[ = \pm \sqrt{\frac{\omega}{c} \left[ 1 - \left( \frac{\ell \pi c}{\omega L_x} \right)^2 - \left( \frac{m \pi c}{\omega L_y} \right)^2 \right]} \]
\[ k_z = \pm \frac{\omega}{c} \left[ 1 - \left( \frac{\ell \pi c}{\omega L_x} \right)^2 - \left( \frac{m \pi c}{\omega L_y} \right)^2 \right]^{\frac{1}{2}} \]

So for this wave, the phase velocity \( c_{\ell m} = \frac{\omega}{k_z} = c \left[ 1 - \left( \frac{\ell \pi c}{\omega L_x} \right)^2 - \left( \frac{m \pi c}{\omega L_y} \right)^2 \right]^{-\frac{1}{2}} \]

Conclude:

i) \( c_{\ell m} > c \), so guided waves travel faster than unguided ones

ii) \( c_{\ell m} \) is undefined when \( 1 - \left( \frac{\ell \pi c}{\omega L_x} \right)^2 - \left( \frac{m \pi c}{\omega L_y} \right)^2 = 0 \)

so there is a cutoff frequency:

\[ \omega_{\text{cutoff}} = \left[ \left( \frac{\ell \pi c}{L_x} \right)^2 + \left( \frac{m \pi c}{L_y} \right)^2 \right]^{\frac{1}{2}} \]

Below this frequency, propagation is attenuated.