I. Motivation for the Dirac Eq.

Try to find an equation that satisfies 2 requirements:

1) $E^2 = p^2 + m^2$ → Its solutions ψ satisfy relativistic energy conservation

Convert these to operators.

$$\underbrace{-\frac{\partial^2 \psi}{\partial t^2} = \left(-\nabla^2 + m^2\right)\psi}$$ "Eq 1"

-but-

2) The equation itself is linear, not quadratic, in $\frac{\partial}{\partial t}$; (so no negative energy solutions)

So it has the general form

$$i\frac{\partial}{\partial t}\psi = \ldots$$

But to be covariant, linear in $\frac{\partial}{\partial t}$ requires also

3) linear in $\vec{\nabla}$, so look for the form:

"Eq 2" $$i\frac{\partial \psi}{\partial t} = \left(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi$$ [Dirac, Proc Royal S A 117, 610 (1928)]

This is the Dirac Eq, with appropriate $\vec{\alpha}$ and β

To find $\vec{\alpha}$ and β, square Eq 2 operators and compare to Eq 1:

$$\left(i\frac{\partial}{\partial t}\right)^2 = (-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi$$

$$= -\sum_{i=1}^{3}\alpha_i^2\frac{\partial^2\psi}{(\partial x^i)^2} - \sum_{\substack{i,j=1\\ i>j}}^{3}(\alpha_i\alpha_j + \alpha_j\alpha_i)\frac{\partial^2\psi}{\partial x^i\partial x^j}$$

$$- im\sum_{i=1}^{3}(\alpha_i\beta + \beta\alpha_i)\frac{\partial\psi}{\partial x^i} + \beta^2 m^2\psi$$

Compare to Eq 1

Eq 1:

$$\left(i\frac{\partial}{\partial t}\right)^2\psi = -\sum_{i=1}^{3}\frac{\partial^2\psi}{(\partial x^i)^2} + m^2\psi$$

Conclude:

$$\alpha_i^2 = 1 \qquad (i=1,2,3)$$

$$\beta^2 = 1$$

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 0 \qquad (i,j=1,2,3;\ i\neq j)$$

$$\alpha_i\beta + \beta\alpha_i = 0 \qquad (i=1,2,3)$$

anti commutation tells us they are not scalars
↓
They are matrices
↓
ψ is a column vector

The solution to this is not unique. The lowest dimension matrices that solve it are

$$\vec{\alpha} = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This is the Dirac-Pauli representation

The σ are the Pauli matrices

Then
$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \text{ etc.}$$

The solutions to the Dirac Eq are 4-component column vectors called <u>Dirac spinors</u>.

II. Dirac γ-matrices (Aitchison p. 172 —)

The Dirac Eq is covariant: its solutions are valid in all frames. But this is hard to see with the $\vec{\alpha}, \beta$ form. It is clearer if we define new matrices γ^i which are functions of α, β.

Plan:

1) Define γ^i matrices
2) Indicate some properties of the γ^i
3) Show how the solutions written with γ^i are clearly covariant.

1) Define:

$$\gamma^0 \equiv \beta = \begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$\gamma^i \equiv \beta\alpha_i$

So $\gamma^1 = \beta\alpha_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix}$

etc.

The $\gamma^2 = \beta\alpha_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$

$\gamma^3 = \beta\alpha_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

Notice different books use different representations.

There is also a fifth one (explain later):

$\gamma^5 \equiv \gamma^0\gamma^1\gamma^2\gamma^3 =$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} & & & 1 \\ & & 1 & \\ & +1 & & \\ +1 & & & \end{pmatrix}\begin{pmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{pmatrix}\begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix} = \begin{pmatrix} 0 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \end{pmatrix},$$

2) Properties of the γ^i

$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \equiv \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ ← This implies $\gamma^0\gamma^k = -\gamma^k\gamma^0$

$(\gamma^0)^{T*} \equiv (\gamma^0)^\dagger = \gamma^0$

$(\gamma^0)^2 = I$ $\qquad g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

$(\gamma^i)^2 = -I \qquad (i = 1, 2, 3)$

$$(\gamma^i)^\dagger = -\gamma^i = \gamma^0\gamma^i\gamma^0 \quad (i=1,2,3)$$

more jargon: for any 4-vector a_μ, $\gamma^\mu a_\mu \equiv \not{a}$

II Facts to note about the Dirac Eq:

Perkins p. 369

1) It is not a single eq (because matrices are involved);

it is a set of 4 simultaneous equations

2) Its solutions, the 4-dimensional Lorentz spinors, have

the property that each of their 4 elements satisfies the KG Eq. (This is H+M Exercise 5.2, worked out in the back).

This is reasonable because KG is the relativistically correct equation for scalar (spinless, one-component) particles.

Name the solns to the Dirac Eq: $\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$

For each ψ_i,

$$(\Box^2 + m^2)\psi_i = 0$$

A free particle solution to this is $\psi_i = u_i(\vec{p})\, e^{-ip\cdot x}$

To see that this soln works, plug in:

$$(\Box^2 + m^2)\psi_i$$

$$\downarrow$$

$$(\Box^2 + m^2)\, u(\vec{p})\, e^{-ip\cdot x}$$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right) u(\vec{p})\, e^{iEt}\, e^{-i\vec{p}\cdot\vec{x}}$$

$$\underbrace{\left[-E^2 - (-\vec{p}^{\,2}) + m^2\right]}_{0} \psi_i$$

But $E^2 = p^2 + m^2$

$= 0$

III. Solving the Dirac Eq. Aitchison p. 68

To find the form of the $u(\vec{p})$

Recall the form of the Dirac Eq without γ-matrices:

$$i\frac{\partial \psi}{\partial t} = (-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi$$

each of these is 2×2

Plug in $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

To make use of the block-diagonal form of $\vec{\alpha}$ and β, write ψ as $\begin{pmatrix} u_A \\ u_B \end{pmatrix}$ each is 2-component

Then $i\frac{d}{dt}\begin{pmatrix} u_A \\ u_B \end{pmatrix} e^{iEt} e^{-i\vec{p}\cdot\vec{x}} = -i\begin{pmatrix} mI & \vec{\sigma}\cdot\vec{\nabla} \\ \vec{\sigma}\cdot\vec{\nabla} & -mI \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} e^{iEt} e^{-i\vec{p}\cdot\vec{x}}$

Operate w/ $\frac{d}{dt}$, Use $\vec{p} = -i\nabla$, so $\nabla = +i\vec{p}$

$$\overset{?}{-}E\begin{pmatrix} u_A \\ u_B \end{pmatrix} e^{\cdots} = \begin{pmatrix} mI & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -mI \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} e^{\cdots}$$

This is 2 coupled equations:

$$E\cdot u_A = m u_A + \vec{\sigma}\cdot\vec{p}\, u_B \longrightarrow (E-m)u_A = \vec{\sigma}\cdot\vec{p}\, u_B \quad \text{Eq } 1$$

$$E u_B = \vec{\sigma}\cdot\vec{p}\, u_A - m u_B \longrightarrow (E+m)u_B = \vec{\sigma}\cdot\vec{p}\, u_A \quad \text{Eq } 2$$

H+M Eq. 5.24

Solve Eq 2 to get $u_B = \frac{\vec{\sigma}\cdot\vec{p}}{(E+m)} u_A$

So a solution to the Dirac Eq. has the form

$$\begin{pmatrix} u_A \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} u_A \end{pmatrix}$$

We can also solve Eq 1 to get $u_A = \frac{\vec{\sigma}\cdot\vec{p}}{(E-m)} u_B$

[illegible] this [illegible] the usual process

So another valid soln to the Dirac Eq has the form:

$$\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{(E-m)} u_B \\ u_B \end{pmatrix}$$

We still have to find either u_A or u_B for each...

IV Find the eigenvectors u_A, u_B and the eigenvalues

This Dirac Eq. is Lorentz invariant – valid in all reference frames, so valid for $\vec{p}=0$. In this frame it is easiest to solve:

$$E u = \begin{pmatrix} mI & 0 \\ 0 & -mI \end{pmatrix} u$$

$$E \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} m & 0 & & \\ 0 & m & & 0 \\ & 0 & -m & 0 \\ & & 0 & -m \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Eigenvalues : $E = \quad +m \quad +m \quad -m \quad -m$

Eigenvectors : $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Concentrate on the positive energy solutions first

We see that there are 2 possible u_A's : $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Call them ——— $\chi^{(1)}$ and $\chi^{(2)}$

So Dirac Eq

Soln #1 is $\begin{pmatrix} u_A \\ u_B \end{pmatrix}^{(1)} = \begin{pmatrix} \chi^{(1)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 0 \end{pmatrix}$ (E'value = m

Soln #2 is $\begin{pmatrix} u_A \\ u_B \end{pmatrix}^{(2)} = \begin{pmatrix} \chi^{(2)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}$ (Also eivalue = m)

Now consider the negative energy solutions (i.e., $E = -m$)

mass is always positive

For these,

Soln # 3 is
$$\begin{pmatrix} u_A \\ u_B \end{pmatrix}^{(3)} = \begin{pmatrix} -\dfrac{\vec{\sigma}\cdot\vec{p}}{E-m}\,\chi^{(2)} \\ \chi^{(2)} \end{pmatrix} = \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{|E|+m} \\ 0 \\ 1 \end{pmatrix}$$

Soln #4 is
$$\begin{pmatrix} u_A \\ u_B \end{pmatrix}^{(4)} = \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E-m}\,\chi^{(1)} \\ \chi^{(1)} \end{pmatrix} = \begin{pmatrix} -\dfrac{\vec{\sigma}\cdot\vec{p}}{|E|+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

-- Normalization still to come --

II Spin is there

Recall

1) If 2 operators commute, they can be measured simultaneously

2) Construct the operator

$$\Sigma\cdot p \equiv \begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix}$$

$\hat{p}$ is the direction of travel of the particle

This commutes with the Dirac Hamiltonian:

$$[H, \vec{\Sigma}\cdot\hat{p}] = \left[\begin{pmatrix} mI & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -mI \end{pmatrix}, \begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix}\right] = 0$$

$\vec{\Sigma}\cdot\hat{p}$ is the spin projection operator, So

1) The difference between Soln 1 and Soln 2 (both have $E = m$) is the spin: up or down

2) The spin projection is an intrinsic quantum # of the particle, naturally described by the Dirac Eq.

Its official name is particle helicity

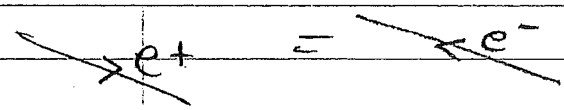
$\overset{s}{\longrightarrow}$ $\overset{p}{\longrightarrow}$ positive helicity

$\overset{s}{\longleftarrow}$ $\overset{p}{\longrightarrow}$ negative helicity

3) Useful relation (see HW) $(\vec{\sigma}\cdot\vec{p})^2 = |\vec{p}|^2$

III. Formalism for converting antiparticles and particles

Recall



$u^{(3)}$ and $^{(4)}$ are "negative energy electrons", i.e.,

$\psi^{(3)} = u^{(3)}(-\vec{p})\, e^{-i(-p)\cdot x}$

It would be useful to have a "positive energy positron version ie, we need to define a "$v(\vec{p})$" such that

$$u^{(3)}(-\vec{p})\, e^{-i(-p)\cdot x} = v^{(2)}(\vec{p})\, e^{ip\cdot x}$$

The Dirac Eq for e^- is

$$(i\partial\!\!\!/ - m)\, u(\vec{p}) = 0$$

Substitute $p = i\partial$

$$(p\!\!\!/ - m)\, u(\vec{p}) = 0$$

For an e^- with $-E, -\vec{p}$ this would be

$$\underbrace{(-p\!\!\!/ - m)}\,\underbrace{u(-\vec{p})} = 0$$

$$\cancel{-}(p\!\!\!/ + m)\, v(\vec{p}) = 0$$

So Dirac Eq for spin-½ antiparticles is

$$\boxed{(p\!\!\!/ + m)\, v(\vec{p}) = 0}$$

The relationship between the v and u solutions is:

$$u_3^{(3)} = \begin{pmatrix} 0 \\ \frac{-\vec{\sigma}\cdot\vec{p}}{|E|+m} \\ 0 \\ 1 \end{pmatrix} \qquad \text{Replace } p^\mu \to -p^\mu \text{ to get } \begin{pmatrix} 0 \\ \frac{+\vec{\sigma}\cdot\vec{p}}{E+m} \\ 0 \\ 1 \end{pmatrix} = v^{(}$$

for negative energy particle

Similarly for $u^{(4)} \to v^{(2)}$

To convert the wavefunction ψ of a particle to the wavefunction ψ_C of its antiparticle, apply this operation:

$$\psi_C = C\gamma_0 \psi^* = C\bar{\psi}^T$$

where $C\gamma_0 = i\gamma^2 = \begin{pmatrix} 0 & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & 0 \end{pmatrix}$

III. Normalization + completeness of u and v

Normalization:

As for KG Eq, let $u^\dagger u = \int \rho\, dV = 2E$

Because there are 4 different orthogonal u's, modify this to

$$u^{i\dagger} u^j = 2E\,\delta_{ij}$$

Similarly $$v^{i\dagger} v^j = 2E\,\delta_{ij}$$

Using N to indicate normalization, for any one of the u^i

$$u^{i\dagger} u^i = |N|^2 \left[1 + \left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)^2\right]$$

$$= \frac{|N|^2 \left(E^2 + m^2 + 2Em + |\vec{p}|^2\right)}{(E+m)^2}$$

Use $E^2 = m^2 + \vec{p}^{\,2}$

$$= \frac{|N|^2 \left(2E^2 + 2Em\right)}{(E+m)^2}$$

Cancel $(E+m)$

$$= \frac{|N|^2\, 2E}{E+m}$$

Magnote p. 59

A general completeness relation for operator A with eigenfunctions ψ_i has the form $\int |\psi|^2 d^3r = \sum_{i=1} |c_i|^2$. Expresses the condition that the ψ_i are sufficient to represent an arbitrary state if $\psi = \sum_{i=1} c_i \psi_i$ 100

Setting $\dfrac{|N|^2 \cdot 2E}{E+m} = 2E$,

$$\boxed{N = \sqrt{E+m}}$$ normalization

Completeness

By explicit calculation one can show

$$\boxed{\begin{aligned} \sum_{i=1}^{2} u^i \bar{u}^i &= \not{p} + m \\ \sum_{i=1}^{2} v^i \bar{v}^i &= \not{p} - m \end{aligned}}$$

completeness; i.e the u's and v's form a basis in which any particular u or v can be expanded

Attention p. 173

NOW

Show the Lorentz invariance (i.e., the Lorentz transformation properties of the matrix elements, the 4-current j, etc.)

Start with Dirac Eq in old form:

$$i\frac{\partial\psi}{\partial t} = -i\vec{\alpha}\cdot\vec{\nabla}\psi + \beta m\psi$$

Mult on left with β

$$\underbrace{i\beta}_{=\gamma^0}\frac{\partial\psi}{\partial t} = -i\underbrace{\beta\vec{\alpha}}_{\vec{\gamma}}\cdot\vec{\nabla}\psi + \underbrace{\beta^2}_{1}m\psi$$

$\downarrow$

Rewrite:

$$i\gamma^0\frac{\partial\psi}{\partial t} = -i\vec{\gamma}\cdot\vec{\nabla}\psi + m\psi$$

$$\downarrow$$

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

$$\downarrow$$

$$(i\not{\partial} - m)\psi = 0$$

Construct the ρ and $\vec{J}$ using the same techniques as for the KG Eq. Recall for the KG Eq we added

(KG Eq) + (KG Eq)*; this gave an eq with the form of the continuity eq: $\frac{\partial}{\partial t}(\text{Something}) + \vec{\nabla}\cdot\left(\begin{matrix}\text{Something}\\ \text{Else}\end{matrix}\right) =$

identify ρ identify $\vec{J}$

Dirac Eq: $(i\gamma^\mu \partial_\mu - m)\psi = 0$ "Eq 1"

$$i\gamma^0 \frac{\partial \psi}{\partial t} + i\gamma^k \frac{\partial \psi}{\partial x^k} - m\psi = 0 \qquad k = 1, 2, 3$$

For the KG Eq we added the complex conjugate. Here because we have matrices we need the Hermitian conjugate $((T)^* = \dagger)$

(Dirac Eq)†: (Remember to reverse order as well as take $*$)

$$-i \frac{\partial \psi^\dagger}{\partial t} \underbrace{\gamma^{0\dagger}}_{\gamma^0} - i \frac{\partial \psi^\dagger}{\partial x^k} \underbrace{\gamma^{k\dagger}}_{-\gamma^k} - m\psi^\dagger = 0$$

$$-i \frac{\partial \psi^\dagger}{\partial t} \gamma^0 - i \frac{\partial}{\partial x^k} \psi^\dagger (-\gamma^k) - m\psi^\dagger = 0$$

To convert to a more useful form mult on right by γ^0

$$-i \frac{\partial}{\partial t} \psi^\dagger \gamma^0 \gamma^0 - i \frac{\partial}{\partial x^k} \psi^\dagger \underbrace{(-\gamma^k \gamma^0)}_{\gamma^0 \gamma^k} - m\psi^\dagger \gamma^0 = 0$$

$$-i \frac{\partial}{\partial t} \psi^\dagger \gamma^0 \gamma^0 - i \frac{\partial}{\partial x^k} \psi^\dagger \gamma^0 \gamma^k - m\psi^\dagger \gamma^0 = 0$$

Define $\overline{\Psi} \equiv \psi^\dagger \gamma^0$ the adjoint

$$-i \frac{\partial}{\partial t} \overline{\Psi} \gamma^0 - i \frac{\partial}{\partial x^k} \overline{\Psi} \gamma^k - m\overline{\Psi} = 0$$

Compactly,

$$\cancel{-}i\,\partial_\mu\bar\Psi\gamma^\mu \cancel{-} m\bar\Psi = 0$$

$$i\,\partial_\mu\bar\Psi\gamma^\mu + m\bar\Psi = 0 \qquad \text{"Eq 2"}$$

Now compute

$$\bar\Psi\cdot(\text{Eq } 1) + (\text{Eq } 2)\cdot\Psi$$

$$\downarrow$$

$$\bar\Psi \cancel{i}\gamma^\mu\partial_\mu\Psi - \cancel{\bar\Psi m \Psi} + \cancel{i}(\partial_\mu\bar\Psi)\gamma^\mu\Psi + \cancel{m\bar\Psi\Psi} = 0$$

$$\bar\Psi\gamma^\mu\partial_\mu\Psi + (\partial_\mu\bar\Psi)\gamma^\mu\Psi = 0$$

This looks like the chain rule for derivatives

$$\downarrow$$

$$\partial_\mu\left[\bar\Psi\gamma^\mu\Psi\right] = 0$$

$$\downarrow$$

Expanding:

$$\frac{\partial}{\partial t}\bar\Psi\gamma^0\Psi - \frac{\partial}{\partial x^k}\bar\Psi\gamma^k\Psi = 0$$

So $\boxed{j^\mu = \bar\Psi\gamma^\mu\Psi}$

with $\boxed{\rho = \bar\Psi\gamma^0\Psi}$ for Dirac Eq.

$\boxed{\vec{J} = \bar\Psi\vec{\gamma}\Psi}$

Rewrite $\partial_\mu j^\mu = 0$; Clearly j^μ is acting as a 4-vector

Why are the γ^μ useful in showing the covariant form?

Write ρ and $\vec{j}$ without them:

Note $\rho = \overline{\Psi}\gamma^0\Psi$

$= \Psi^\dagger \gamma^0\gamma^0\Psi$

$= \Psi^\dagger\Psi$ (still preserves the form of $\Psi^*\Psi$ but with no explicit indices it's hard to see that this is the 0th element of the 4-vector)

Notice $\rho = \overline{\Psi}\gamma^0\Psi = \Psi^\dagger\Psi = \sum_{i=1}^{4} |\Psi_i|^2 \geq 0$ as necessary for meaningful probabilities

To describe the current density rather than probability density, we still just multiply j^μ by $-e$:

$$j^\mu_{current} = -e j^\mu_{prob} = -e\overline{\Psi}\gamma^\mu\Psi$$

I Bilinear covariants

How are these ψ, $\bar{\psi}$, etc related to the wavefunctions for physical particles?

Need to combine $\psi, \bar{\psi}, \gamma$'s in a way that is

① Lorentz covariant

② produces a quantity with the response to parity, boost that we observe in the laboratory

The old $\psi^*\psi$ or $\psi^\dagger\psi$ is not covariant by itself, so we need some γ's.

There are 16 linearly independent combinations of the form

$\bar{\psi}_i$ (some combination of γ's) $\cdot \psi_j$

These are

$\bar{\psi}\psi$	1
$\bar{\psi}\gamma^5\psi$	1
$\bar{\psi}\gamma^\mu\psi$	4
$\bar{\psi}\gamma^\mu\gamma^5\psi$	4
$\bar{\psi}\sigma^{\mu\nu}\psi$, where $\sigma^{\mu\nu} \equiv \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$	6

Inserting additional γ's create products that can be reduced by known identities to one of these 16.

So these 16 form a basis for all 4×4 matrices

Why this is important for physics:

Bjorken + Drell p. 26

Each of the 5 classes has unique Lorentz transformation properties

Kinds of transformations	Effect	Operator
Parity S_p:	$\psi(\vec{x},t) \rightarrow \psi'(-\vec{x},t)$	γ^0 (i.e. $\psi' = \gamma^0 \psi$)
Boost S_L:	$\psi \rightarrow \psi' = S_L \psi$	$1 - \frac{i}{4}\sigma_{\mu\nu}\varepsilon^{\mu\nu}$

we can also write this as
$(x^\nu)' = a^\nu_\mu x^\mu$

where the a's indicate the usual Lorentz trans. coefficients

Bilinear	Apply S_L to it, get	Apply S_p	So it is a:
$\bar\Psi 1 \Psi$	$\bar\Psi \underbrace{S_L^{-1}(1) S_L} \Psi = \bar\Psi\Psi$		scalar
$\bar\Psi \gamma^\mu \Psi$	$\bar\Psi S_L^{-1}\gamma^\mu S_L \Psi = a^\nu_\mu \bar\Psi \gamma^\mu \Psi$		vector
$\bar\Psi \gamma^5 \Psi$	$\det\lvert a\rvert \bar\Psi \gamma^5 \Psi$	$-\bar\Psi \gamma^5 \Psi$	pseudoscalar
$\bar\Psi \gamma^5 \gamma^\nu \Psi$	$\det\lvert a\rvert a^\nu_\mu \bar\Psi \gamma^5 \gamma^\nu \Psi$	$-\bar\Psi \gamma^5 \gamma^\nu \Psi$	pseudovector
$\bar\Psi \sigma^{\mu\nu} \Psi$	$a^\mu_\alpha a^\nu_\beta \bar\Psi \sigma^{\alpha\beta} \Psi$		tensor

Jayson:
Any interaction (E+M, weak, etc) can be written in terms of these, so different interactions manifest conservation laws, or violations (e.g., parity) on the basis of this structure