For an e^- around a nucleus

$\vec{m} \to$ nuclear mag. mom

$\vec{B} \to$ mag field generated by e^-, eval @ nucleus

Recall classical $\vec{m} = \frac{e\vec{L}}{2Mc}$ where $\vec{L}$ is orbital ang. mom, from a current loop model

Similarly B is derived from a current

$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{J \times \hat{R}}{R^2}\,d\text{vol}$ again $\vec{J} \to \vec{I}$

Gasiorowicz + Rosner speculated that one could replace $\vec{L} \to$ Spin $\vec{S}$.

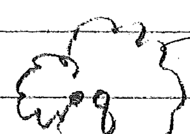
Good agreement between this coarse model and measured masses: all light mesons to < 2%

(Griffiths Table 5.3, p.180)

Note: the m_i are constituent quark masses, see below.

→ To understand the quark mass terms:

Recall the anti-screening of quarks by gluons



temporarily removes some color

so quark color charge is smaller to probes that penetrate deeper.

Distinguish a "bare" quark from a "dressed" quark

with gluons

input to QCD calc

also called "constituent", extracted from bound state measurements

	bare quark masses	constituent (covered) masses
u	~2 MeV	308 MeV
d	~5 MeV	308 "
c	1 GeV	
s	0.1 GeV	483 MeV
t	174 GeV	
b	4.1 GeV	

How does the bare quark acquire mass from the cloud of massless gluons?

For comparison:

Consider the hydrogen atom: its mass is <u>less</u> than the sum of its components by the amt of work done in system (EM interaction) to form the bound state:

$$m_H = m_p + m_e - 13.6\,eV$$

This work represents what is required to move the constituents to a point (∞) at which their interaction is zero.

Now consider a strongly interacting bound state. <u>Because</u> of asymptotic freedom (quarks are non-interacting when the distance between them is zero) the work term, W_{strong}, is <u>positive</u>.

What sets the scale of W_{strong} at ~300 MeV?

QCD has a parameter Λ that characterizes the regime in which α_s transitions from being "small" (suitable for perturbative expansion) to "large" (series conversion is not guaranteed).

$$\alpha_s(Q^2) = \frac{12\pi}{(33-2n_f)\log\left(\frac{Q^2}{\Lambda^2}\right)}$$

momentum transfer

quark flavors with masses below Q

$\Lambda \sim 300$ MeV (see Griffiths p.301)

• Chapter 6 - Feynman Rules

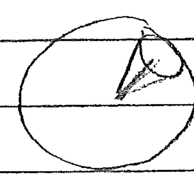
We typically want to predict things that can be measured in data, e.g.

events / time

(i) differential cross section $\frac{d\sigma}{d\Omega} = \frac{dN}{\mathcal{L}\, d\Omega}$

solid angle (str)

luminosity = $\frac{\text{events}}{\text{cm}^2 \cdot \text{sec}}$



Result has units $\frac{1}{\text{cm}^2}$.

Characterizes Prob of interaction producing tracks in $d\Omega$

for fixed cone opening angle, ratio of cap area to sphere area.

Recall from classical mechanics that a measurement of cross section gives info about the potential:

$$E = \frac{\mu \dot{r}^2}{2} + \frac{l^2}{2\mu r^2} + V(r)$$

$$d\theta = \frac{d\theta}{dt} \cdot \frac{dt}{dr} \cdot dr$$

$$\dot{\theta} = \frac{l}{\mu r^2} \qquad \dot{r} = \pm\left[\frac{2}{\mu}\left(E - U - \frac{l^2}{2\mu r^2}\right)\right]^{1/2}$$

So $$\theta = \pm \int \frac{(l/r^2)\,dr}{\left[2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)\right]^{1/2}}$$

$$\vec{l} = \vec{r} \times \vec{p} \rightarrow b\,\mu v$$

$$\sigma = \frac{b}{\sin\theta} \frac{db}{d\theta}$$

(ii) lifetime. If there are n channels (ways) for a particle to decay, and each has rate Γ then # particles remaining after t = $N(t) = N_0 e^{-\Gamma t}$ ↑ decay width

$$\Gamma_{TOT} = \sum_n \Gamma_n$$

Then $\tau = \frac{1}{\Gamma_{TOT}}$

Def: Branching ratio: Normalized probability for decay into channel i = $\frac{\Gamma_i}{\Gamma_{TOT}}$

The formula for decay rates follows from QM transition probabilities in non-relativistic perturbation theory:

Consider ϕ_n, the solutions to the free particle Schroedinger Eq:

$$H_0 \phi_n = E_n \phi_n$$

They are orthonormal: $\int \phi_m^* \phi_n \, dvol = \delta_{mn}$

So they form a basis for expressing any other solution. How are they modified by introduction of a potential $V(x,t)$?

$$[H_0 + V(x,t)]\Psi = i\frac{d\Psi}{dt} \qquad \hbar = 1 \qquad \text{Eq 1}$$

Apply separation of variables. Get $\Psi = \sum_n \Psi_n(x) e^{-iE_n t}$.

Write Ψ_n in the φ basis?

$$\Psi = \sum_n a_n(t)\, \varphi_n(x)\, e^{-iE_n t} \qquad \text{Eq 2}$$

To find the a_n, plug Eq 2 into Eq 1

$$[H_0 + V]\sum_n \underbrace{a_n}_{f(t)} \underbrace{\varphi_n}_{f(x)} e^{-iE_n t} = i\frac{d}{dt}\left[\sum_n a_n \varphi_n e^{-iE_n t}\right]$$

$$\sum H_0 a_n \varphi_n e^{-iE_n t} + \sum V a_n \varphi_n e^{-iE_n t} = i\sum \varphi_n \frac{d}{dt}\left[a_n e^{-iE_n t}\right]$$

$$\cancel{\sum E_n a_n \varphi_n e^{-iE_n t}} + \sum V a_n \varphi_n e^{-iE_n t}$$

$$= i\sum \varphi_n \left\{\cancel{a_n(-iE_n)e^{-iE_n t}} + \frac{da_n}{dt} e^{-iE_n t}\right\}$$

We get

$$\sum_n V(x,t)\, a_n \varphi_n e^{-iE_n t} = i\sum_n \varphi_n \frac{da_n}{dt} e^{-iE_n t}$$

Mult both sides of φ_f^* and integrate over $\int d\text{Vol}$

$$\int d\mathrm{Vol}\, \varphi_f^* \sum_n V(x,t)\, a_n e^{-iE_n t} = \int d\mathrm{Vol}\, i\, \underbrace{\varphi_f^* \sum_n \varphi_n}_{\delta_{fn}} \frac{da_n}{dt} e^{-iE_n t}$$

$$= i \sum_n \delta_{fn} \frac{da_n}{dt} e^{-iE_n t}$$

$$= i \frac{da_f}{dt} e^{-iE_f t}$$

Mult. both sides by $-ie^{iE_f t}$:

$$\frac{da_f}{dt} = -i \sum_n a_n \int d\mathrm{Vol}\, \varphi_f^* V \varphi_n\, e^{i(E_f - E_n)t}$$

Suppose that before V acts, the initial state is a single term in the series, so

@ $t = -\frac{T}{2}$ $\quad a_i = 1$
$\quad a_{i \neq n} = 0$

Then $\frac{da_f}{dt} = -i \int d\mathrm{Vol}\, \varphi_f^* V \varphi_i\, e^{i(E_f - E_i)t}$

$$a_f = -i \int_{-T/2}^{t} dt' \int d\mathrm{Vol}\, \varphi_f^* V \varphi_i\, e^{i(E_f - E_i)t}$$

Consider a specific $t = T/2$

$$a_f(T/2) = -i\int_{-T/2}^{+T/2} dt \int d^3x \left[\varphi_f(x) e^{-iE_f t}\right]^* V(x,t) \left[\varphi_i e^{-iE_i t}\right]$$

Call this T_{fi}, the amplitude for transition from i to f due to V, during interval T.

Suppose $V = V(x, \text{NO } t)$

Then $a_f = T_{fi} = -i \underbrace{\int d^3x\, \varphi_f V \varphi_i}_{\text{call this "}V_{fi}\text{"}} \underbrace{\int_{-\infty}^{+\infty} dt\, e^{i(E_f - E_i)t}}_{2\pi\delta(E_f - E_i)}$

Probability of transition $= |a_f|^2$

Rate of transition $= \Gamma = \lim_{T\to\infty} \dfrac{|T_{fi}|^2}{T}$

$$= \lim_{T\to\infty} \frac{1}{T} |V_{fi}|^2 |-i|^2 \cdot 2\pi \underbrace{\delta(E_f - E_i) \int_{-T/2}^{T/2} dt\, e^{i(E_f - E_i)t}}$$

$$\cancel{\frac{1}{T}} |V_{fi}|^2 \cdot 1 \cdot 2\pi\, \delta(E_f - E_i) \cancel{\int_{-T/2}^{+T/2} dt}$$

$$= 2\pi |V_{fi}|^2 \delta(E_f - E_i)$$

This has physical meaning when applied to transitions between a particular initial state and final state

Let ρ = density of final states.
So $\rho(E)dE$ = # final states in range E_f to $E_f + dE_f$

The $\Gamma_{fi} = 2\pi \int dE_f \, \rho(E_f) |V_{fi}|^2 \delta(E_f - E_i)$

specific

Now understand what ρ is:

Begin with relativistic energy conservation:

$$E^2 = p^2 + m^2$$

Substitute operators $E \to i\hbar \frac{\partial}{\partial t}$

$p \to -i\hbar \nabla$ let $\hbar = 1$

Act on wavefunction ϕ:

$$-\frac{\partial^2 \phi}{\partial t^2} = -\nabla^2 \phi + m^2 \phi$$

$$-\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = m^2 \phi$$

The Klein-Gordon Eq. "The relativistic Schroedinger Eq. for Spin-0 particles"

Compute $[-i\phi^*(KG)] - [-i\phi(KG)^*]$:

$$\frac{\partial}{\partial t}\left[i\left(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t}\right)\right]+\nabla\cdot\left[-i\left(\phi^*\nabla\phi-\phi\nabla\phi^*\right)\right]=0$$

Compare to the continuity eq:

$$\frac{\partial\rho}{\partial t}+\nabla\vec{J}=0$$

We see that $\rho = i\left(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t}\right)$

Suppose ϕ is a plane wave: $\phi = N e^{i(\vec{p}\cdot\vec{x}-Et)}$

Then $\rho = 2E|N|^2$ compensates for Lorentz contraction of the volume to keep $\rho\, d^3x$ constant.

Normalize a unit volume:

$$\int \rho\, dV = 2E \text{ particles}$$

Then $N = \frac{1}{\sqrt{V}}$

So $\rho = \frac{2E}{V}$

Choose unit volume: $V \to 1$

$\rho = 2E$ particles per QM state

How many QM states are available to the process?
Recall $\Delta p \Delta x \sim h = 2\pi\hbar$, $\hbar \to 1$

In 3D: $\frac{d^3p \cdot V}{(2\pi)^3}$ = # QM phase space cells per 2E particles

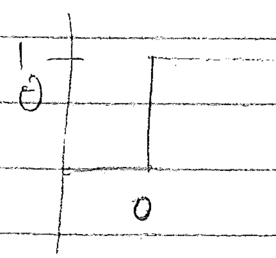
So $\frac{d^3p \cdot V}{(2\pi)^3 \cdot 2E}$ = # cells per particle

Consider a decay at rest, 1 particle produces n products
$E = m_1 c^2_{\to 1}$. So $\frac{1}{2E} \to \frac{1}{2m_1}$

If s products are identical, their exchange should not count as a separate state, so include a correction factor $\frac{1}{s!}$ for permutations.

Assume energy conservation $(2\pi)\delta(p_j^2 - m_j^2c^2)$ for each final state

Assume $E \geq 0$ $\Theta(p_j^0)$



Integrate over all possible combinations of final state E/p.
(Pick up a $\int d^4 p_j$)
Generalize from 3-D to 4-D; $2\pi\delta(E_f - E_i) \to (2\pi)^4\delta^4(p_1 - p_2 - \cdots p_n)$

$$\frac{d^3p}{(2\pi)^3} \to \frac{d^4p}{(2\pi)^4}$$

$\Gamma \rightarrow \Gamma \cdot$ density of final states

$$= \int \prod_j \frac{(2\pi)}{(\hbar)} |V_{fi}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - \dots p_n) \, \delta(p_j - m_j c^2) \, \Theta(p_j^0) \cdot \frac{d^4 p}{(2\pi)^4} \cdot \frac{1}{2m_1} \cdot S$$

had been set = 1 in Eq 1.

For a decay to 2-body final state this becomes

$$\Gamma = \frac{S |\vec{p}| |V_{fi}|^2}{8\pi\hbar m_1^2 c}$$

$|\vec{p}|$ is the momentum of either outgoing particle

For a scatter it makes more sense to talk about a cross section instead of rate

$$\sigma \sim \frac{\text{rate}}{\text{initial flux}}$$

For $1 + 2 \rightarrow 3 + 4 + \dots + n$ this leads to

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S |V_{fi}|^2}{(E_1 + E_2)^2} \frac{|\vec{p}_f|}{|\vec{p}_i|} \quad \text{in COM}$$

Note V_{fi} is often called M, the matrix element

Recipe for assembling M: Assume scalar particle ("Toy Theory")

(0) Draw the picture.

(i) Label the momenta (p_i) including direction, for every line. Internal momenta are q_i.
Note the momenta lines do not necessarily point in the same direction as the "fermion" lines

(ii) for every vertex write $-ig$

(iii) for every internal line write $\frac{i}{q_j^2 - m_j^2 c^2}$

(iv) for every vertex write $(2\pi)^4 \delta^4(k_1 + k_2 + k_3)$
where k_i is incoming.
For outgoing use $-k_i$

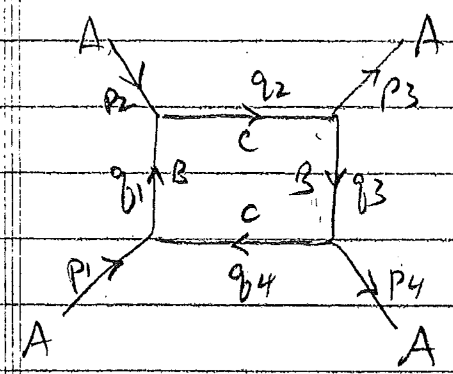
(v) For every internal line write $\frac{1}{(2\pi)^4} d^4 q_j$ and integrate over q_j

After completing the integration, replace
$(2\pi)^4 \delta(\quad) \rightarrow i$

The result is M

SC

Example for a spinless particle interaction



$$\iiiint (-ig)^4 \cdot \frac{i}{q_1^2 - m_B^2c^2} \cdot \frac{i}{q_2^2 - m_C^2c^2} \cdot \frac{i}{q_3^2 - m_B^2c^2} \cdot \frac{i}{q_4^2 - m_C^2c^2} \cdot (2\pi)^4$$

$$\cdot\, \delta^4(p_1 + q_4 - q_1) \cdot (2\pi)^4 \delta^4(p_2 + q_1 - q_2) \cdot (2\pi)^4 \delta^4(q_2 - q_3 - p_3)$$

$$\cdot\, (2\pi)^4 \delta^4(q_3 - q_4 - p_4) \cdot \frac{d^4q_1}{(2\pi)^4} \cdot \frac{d^4q_2}{(2\pi)^4} \cdot \frac{d^4q_3}{(2\pi)^4} \cdot \frac{d^4q_4}{(2\pi)^4}$$

$$= g^4 \iiiint \frac{\delta^4(p_1 + q_4 - q_1)\, \delta^4(p_2 + q_1 - q_2)\, \delta^4(q_2 - q_3 - p_3)}{q_1^2\, q_2^2\, q_3^2}$$

$$\cdot\, \frac{\delta^4(q_3 - q_4 - p_4)\, d^4q_1\, d^4q_2\, d^4q_3\, d^4q_4}{q_4^2}$$

Do the q_4 integral $(q_4 \to q_1 - p_1)$
and the q_3 integral $(q_3 \to q_2 - p_3)$. We get:

$$g^4 \iint \frac{\delta^4(p_2 + q_1 - q_2)\, \delta^4(q_2 - p_3 - q_1 + p_1 - p_4)\, d^4q_1\, d^4q_2}{(q_1 - p_1)^2 (q_2 - p_3)^2\, q_1^2\, q_2^2}$$

Do the q_2 integral ($q_2 \to p_2 + q_1$). We get

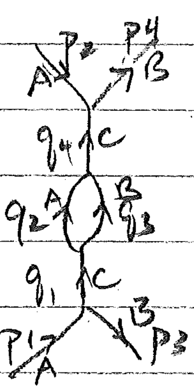
$$g^4 \int \frac{\delta^4(p_2 + \cancel{q_1} - p_3 - p_4 - \cancel{q_1} + p_1)\, d^4 q_1}{q_1^2 (p_2+q_1)^2 (p_2+q_1-p_3)^2 (q_1-p_1)^2}$$

Cancel $(2\pi)^4 \delta^4(p_2+p_1-p_3-p_4)$ and multiply by i.
Drop the subscript on q_1:

$$\mathcal{M} = \frac{i g^4}{(2\pi)^4} \int \frac{d^4 q}{q^2 (p_2+q)^2 (p_2+q-p_3)^2 (q-p_1)^2}$$

- Loops lead to infinities, require renormalization.

Consider a line that includes a loop



To construct the matrix element $\mathcal{M}$, apply:

4 vertices (pre-cancelling all the $(2\pi)^4$ factors)

$$g^4 \iiiint \frac{\delta^4(p_1-q_1-p_3)\,\delta^4(q_1-q_2-q_3)\,\delta^4(q_2+q_3-q_4)}{(q_1^2-m_C^2c^2)(q_2^2-m_A^2c^2)(q_3^2-m_B^2c^2)} \cdot$$

$$\frac{\delta^4(q_4+p_2-p_4)}{(q_4^2-m_C^2c^2)} \cdot d^4q_1\, d^4q_2\, d^4q_3\, d^4q_4$$

Integrate over q_1 $(q_1 \to p_1-p_3)$
" " q_4 $(q_4 \to p_4-p_2)$
" " q_2 $(q_2 \to p_1-p_3-q_3)$

What is left:

$$g^4 \int \frac{\delta^4(p_1+p_2-p_3-p_4)\, d^4q_3}{[(p_1-p_3)^2-m_C^2c^2]^2[(p_1-p_3-q_3)^2-m_A^2c^2][q_3^2-m_B^2c^2]}$$

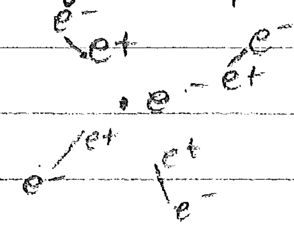
To convert this to $\mathcal{M}$,
- drop the $\delta^4(\quad)$
- multiply by $\frac{i}{(2\pi)^4}$
- let $q_3 \to q$

$$\mathcal{M} = \frac{i g^4}{(2\pi)^4} \int \frac{\overbrace{d^4q}^{q^3\,dq}}{[\quad]\underbrace{[(\cdots - q)^2 \cdots][q^2 - \cdots]}_{q^4}}$$

no limit to loop momenta

$$\mathcal{M} \sim \int_0^{\infty} \frac{dq}{q} \to \ln q \Big|^{\infty} \to \infty$$

Interpreting this ∞ requires renormalization. (Ch 7)
in which the ∞ is absorbed into the "g"
maybe g really is infinite when unscreened.
Recall



Ch 7. - Dirac Eq
Recall the Klein-Gordon Eq:

$$-\frac{\partial^2 \psi}{\partial t^2} + \nabla^2 \psi = m^2 \psi$$

* why this equation bothered people:

(1) Solns: $\psi = Ne^{-ip\cdot x} = Ne^{+i\vec{p}\cdot\vec{x} - iEt}$

Take $\frac{\partial^2 \psi}{\partial t^2}$ and $\nabla^2 \psi$, plug into KG Eq, get

$$E = \pm\sqrt{|\vec{p}|^2 + m^2}$$

How to interpret the negative values? It appears that an infinite set of increasingly lower neg values is availal so there is no ground state

(2) Same procedure to get ρ and $\vec{J}$ gives

$$\rho = i\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) = 2E|N|^2$$

*negative solns

$$\vec{J} = -i\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right) = 2\vec{p}\,|N|^2$$

So the probability density ρ appears to have negative solutions,

(3) It does not naturally include spin which was known from the Stern Gerlach expt in 1922

(compare Schroedinger Eq: 1926
KG 1927
Dirac 1928)

How to address these problems:

① Combine ρ and $\vec{J}$ to make the 4-vector j^μ:

$$j^\mu = (\rho, \vec{J}) = i\left(\psi^* \partial^\mu \psi - \psi \partial^\mu \psi^*\right) = 2|N|^2 (E, \vec{p})$$

Try multiplying this by $-e$ and interpreting it as physical charge density + current rather than prob. density + current.

So $j^\mu \rightarrow -e j^\mu = -2e|N|^2(E, \vec{p})$

② Call this the 4-current of an electron:

$$j^\mu(e^-, E, \vec{p}) = -2e|N|^2(E, \vec{p})$$

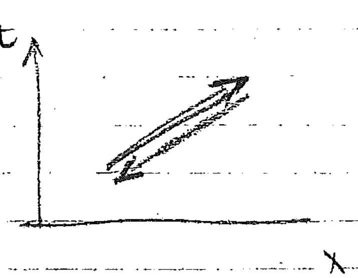
Compare to the 4-current of a positron with same energy, momentum $\vec{p}$:

$j^\mu(e^+, E, \vec{p}) = +2e|N|^2(E, \vec{p})$ ← emission ($\vec{p}$) of e^+ with E

$= -2e|N|^2(-E, -\vec{p})$ ← absorption ($-\vec{p}$) of e^- with $-$

emission: object travels forward in time

absorption: " " backward " " (i.e., maintain same path, reverse direction of $\vec{p}$)



So interpret the negative-energy electrons with $(-p)$ (ie travelling backward in time) as positive-energy positrons with $(+p)$ (ie travelling forward in time.)

③ This does not naturally address the inclusion of spin. For that we need a different eq. (Dirac Eq)

To prepare for this, read about four-vector notation (Griffiths section 3.2) ignoring Lorentz transformations

Note

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

$$x_\mu = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z)$$

$$x^\mu = g^{\mu\nu} x_\nu \quad \text{where } g^{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

implied sum over ν
if one is super, other is sub

Note $g^{\mu\nu} = (g_{\mu\nu})^{-1} = g_{\mu\nu}$