23

SYSTEMS OF UNITS — A GUIDE FOR THE PERPLEXED

We have been using exclusively SI units which, for our purposes, are the same as those of the rationalized MKSA system, and we will continue to do so. This means that our unit system is based on *four* arbitrarily chosen and defined quantities—the meter, kilogram, second, and ampere. Primarily because of historical reasons based on the originally separate development of electricity and magnetism, other systems of units have been used and continue to be used. This is particularly the case in more advanced treatments of physics, especially in subjects like quantum mechanics and its application to the microscopic properties of matter. As a result, a student whose training has been conducted entirely in terms of MKSA units very often faces the problem of answering two questions: In what system of units is *this* equation written? What *numbers* do I put in it in order to work this problem? This chapter provides some guidance in how to answer these questions. Consequently, we do not give an exhaustive discussion of various unit systems but primarily try to indicate how they originated, the effects on the forms of our basic equations, and what to do about it. Thus, in a sense, this short chapter is a digression, but, since we have just finished the task of writing electromagnetism in its most general and fundamental form, this is a useful point at which to consider these questions.

23-1 ORIGIN OF OTHER SYSTEMS OF UNITS

In order to see how different systems can occur, it is sufficient for our purposes to consider two basic experimental results—one electrostatic and one magnetostatic. From Coulomb's law (2-3), we know that the magnitude of the force between two point charges has the form

$$F = C_e \frac{qq'}{R^2} \tag{23-1}$$

where C_e is a constant of proportionality whose numerical value will depend on the units that are used. We previously chose to write $C_e = 1/4\pi\epsilon_0$. Similarly, the magnitude of the force per unit length between two parallel currents as found from Ampère's law is given by (13-13) and (13-14) and can be written as

$$\frac{dF}{dz} = 2C_m \frac{II'}{\rho} \tag{23-2}$$

where C_m is another constant of proportionality that we previously wrote as $C_m = \mu_0/4\pi$. In addition, all systems of units use the definition of current given by (12-2): $I = dq/dt$.

If one always uses the same set of mechanical units in them, then the two forces involved will have the same dimensions and we see that the combinations $C_e qq'/R^2$

and $C_m II'$ also must have the same dimensions so that the ratio

$$\frac{C_e}{C_m} = c^2 \tag{23-3}$$

must have the dimensions of (distance/time)^2, that is, c has the dimensions of a *speed*. The value of this ratio has been measured many times and the *experimental result* is that

$$c = 3 \times 10^8 \text{ meters/second} \tag{23-4}$$

which is the same as the measured speed of light in a vacuum. The best value of c as presently known is 2.99792458×10^8 meters/second, but the value given in (23-4) is accurate enough for us here. (As we will see in the next chapter, the agreement between this ratio and the speed of light is not accidental.) We have already taken this numerical result into account in the values we gave for C_e and C_m and we find that $C_e/C_m = (4\pi\epsilon_0)^{-1}/(\mu_0/4\pi) = (\mu_0\epsilon_0)^{-1} = (9 \times 10^9)/(10^{-7}) = (3 \times 10^8 \text{ meters/second})^2$ with the use of (2-5) and (13-2); this is in agreement with (23-3) and (23-4). In other words, we have the fundamental result that

$$\mu_0\epsilon_0 = \frac{1}{c^2} \tag{23-5}$$

for the MKSA system that we are using.

The various systems of units used in electromagnetism essentially differ in the way in which these constants are chosen. It is clear that either C_e or C_m can be chosen arbitrarily, but then the value of the other is fixed by the requirement of (23-3).

All other systems of any interest are based on the use of the CGS system in which everything is expressed in terms of the arbitrary choice of *three* fundamental units for length, mass, and time; these are, respectively, the centimeter, gram, and second. The mechanical units are then found in the usual way from their definitions. Thus, the force unit is 1 gram × 1 centimeter/(second)^2 = 1 dyne. The unit of work or energy will be the product of a unit force and a unit displacement: 1 dyne-centimeter = 1 erg. The unit of power will be 1 erg/second, and so on.

Another distinction that is made between unit systems concerns whether they are "rationalized" or "unrationalized." What this means, in effect, is that for a rationalized system there are no numerical factors of 4π appearing in Maxwell's equations, while, as we will see, they do appear if an unrationalized system is used. Equations 21-19 through 21-22 show that we are using rationalized MKSA units. (The use of a rationalized system does not make 4π disappear, rather it simply means that 4π's are found elsewhere in results found from Maxwell's equations; thus, the choice of which type to use is somewhat a matter of taste.)

23-2 THE ELECTROSTATIC AND ELECTROMAGNETIC SYSTEMS

Suppose that you felt that Coulomb's law was a fundamental result that was the best place to start in defining a system of units for electromagnetism. Your natural inclination would be to give this equation as simple an appearance as possible; this can certainly be done by *choosing* $C_e = 1$. Then, from (23-3), C_m *must* be taken to be $1/c^2$ and (23-1) and (23-2) would become

$$F = \frac{qq'}{R^2} \qquad \frac{dF}{dz} = \frac{2II'}{c^2\rho} \qquad \text{(esu)} \tag{23-6}$$

This procedure leads one to the *electrostatic system of units* (esu). We see from the first expression in (23-6) that two equal unit charges a distance 1 centimeter apart will repel each other with a force of 1 dyne; the unit of charge defined in this way is called a *statcoulomb* (from electro*stat*ic). The unit of current will then be 1 statcoulomb/second = 1 statampere, and that of potential difference 1 erg/statcoulomb = 1 statvolt. One can continue this process and define a statfarad, statohm, and so on, and in this way develop a consistent and complete description. At some point, however, one has to decide how to define **B** and relate it to **E**; this is done in this system by writing Faraday's law as $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ or, equivalently, the Lorentz force as $q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. In this pure form, however, this system is seldom used anymore and we will not describe it further. Nevertheless, it should be pointed out that it is very common to find quantities measured in this system but *not* given in statamperes, statfarads, and so on, but merely stated as being so many "electrostatic units" or just so many "esu."

Now suppose that you were more interested in and had more experience with magnetostatics than in electrostatics; then you might feel that the expression (23-2) is a better starting point and you would like to simplify it as much as possible. This can be done by choosing $C_m = 1$ so that $C_e = c^2$ and (23-1) and (23-2) become

$$F = \frac{c^2 qq'}{R^2} \qquad \frac{dF}{dz} = \frac{2II'}{\rho} \qquad \text{(emu)} \tag{23-7}$$

Such a procedure leads to the *electromagnetic system of units* (emu). We now see from the second expression in (23-7) that two very long equal parallel unit currents 1 centimeter apart will attract each other with a force of 2 dynes/centimeter; the unit of current defined in this way is called an *abampere* (from *ab*solute). The unit charge will be 1 abcoulomb = 1 abampere-second and one can continue this process to get abvolts, abfarads, and the like; it is also very common to use simply the terminology of "electromagnetic units" or "emu." Again the definitions of **E** and **B** are related by writing $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ or $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. In this pure form, the electromagnetic system is practically never used. What *is* still very much used, however, and the system which one needs to be able to deal with, is that which we consider next.

23-3 THE GAUSSIAN SYSTEM

This is an *unrationalized CGS* system that is *mixed* in the sense that electric quantities are measured in electrostatic units while magnetic quantities are measured in electromagnetic units. For our purposes, it will suffice simply to quote the form that the basic equations assume for this system. Maxwell's equations in general are

$$\begin{gathered} \nabla \cdot \mathbf{D} = 4\pi\rho_f \qquad \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J}_f + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \end{gathered} \tag{23-8}$$

where the various field vectors are related by

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} \qquad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} \tag{23-9}$$

and the Lorentz force is

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \tag{23-10}$$

[It follows from (23-8) that the equation of continuity still has the form $\nabla \cdot \mathbf{J}_f + (\partial \rho_f/\partial t) = 0$.]

Where they are applicable, the various constitutive equations are written

$$\mathbf{D} = \epsilon\mathbf{E} \qquad \mathbf{H} = \mathbf{B}/\mu \qquad \mathbf{J}_f = \sigma\mathbf{E}$$
$$\mathbf{P} = \chi_e\mathbf{E} \qquad \mathbf{M} = \chi_m\mathbf{H} \tag{23-11}$$

so that

$$\epsilon = 1 + 4\pi\chi_e \qquad \mu = 1 + 4\pi\chi_m \tag{23-12}$$

Expressions involving the potentials are easily seen to become

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{23-13}$$

while energy formulas of interest are

$$u = \frac{1}{8\pi}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}) \qquad \mathbf{S} = \frac{c}{4\pi}(\mathbf{E}\times\mathbf{H}) \tag{23-14}$$

where the first holds for linear media.

We see from the above that all of the field vectors **E**, **D**, **B**, **H**, **P**, and **M** have the *same dimensions*; this, of course, has not kept people from giving different *names* to the *units*. This is most prevalent with respect to magnetic quantities and the common usage is as follows: **B**, gauss; **H**, oersted; **M**, oersted (but see the next section); Φ, 1 gauss-(centimeter)^2 = 1 maxwell.

It also follows from (23-9) that in a vacuum, **D** = **E** and **H** = **B**. The quantities ϵ, μ, χ_e, and χ_m are all dimensionless; we discuss their numerical values in the next section.

Furthermore, it is not uncommon to find a "modified" Gaussian system being used. It is just like the one we have described *except* that, while charge is still measured in statcoulombs (esu), current is measured in abamperes (emu). What this does, in effect, is to replace any symbol for current by c times that symbol (e.g., $\mathbf{J}_f \to c\mathbf{J}_f$). The only one of Maxwell's equations that is affected by this is Ampère's law, and then the equation of continuity, which become

$$\nabla \times \mathbf{H} = 4\pi\mathbf{J}_f + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \qquad \nabla\cdot\mathbf{J}_f + \frac{1}{c}\frac{\partial \rho_f}{\partial t} = 0 \tag{23-15}$$

Finally, the *Heaviside-Lorentz* system is simply a rationalized Gaussian system; when this is used, the effect is to replace every 4π in the equations (23-8) through (23-12) by unity; for example, one gets $\nabla\cdot\mathbf{D} = \rho_f$ and $\mathbf{D} = \mathbf{E} + \mathbf{P}$. The factors of c still remain, however.

If a particular author does not state the unit system that is being used, one can generally deduce what it is by looking at the form of some familiar results, preferably Maxwell's equations.

23-4 HOW TO COPE WITH THE GAUSSIAN SYSTEM

In principle, any desired result in the Gaussian system can be derived by starting with Maxwell's equations (23-8) and using the expressions (23-9) through (23-13) as required. This is not always convenient and it is desirable to have a method that will enable one to transform a given result written in the Gaussian system into the corresponding one in the MKSA system and *vice versa*. Table 23-1 provides a recipe for doing this. In order to use this table, one replaces a symbol in the column labeled by the system in which the formula is written by the symbol or combination listed for the other system. Symbols representing essentially mechanical quantities are unchanged.

Table 23-1. Conversion of symbols in equations
Symbols representing essentially mechanical quantities (length, mass, time, force, work, energy, power, etc.) are not changed (nor are derivatives). To convert an equation written in the MKSA system to the corresponding one in the Gaussian system, replace the symbol listed under the column labeled MKSA by that listed under Gaussian. The entries can also be used to convert a Gaussian equation to an MKSA one by going from right to left in the table.

Quantity	MKSA	Gaussian
Capacitance	C	$4\pi\epsilon_0 C$
Charge	q	$(4\pi\epsilon_0)^{1/2}q$
Charge density	$\rho, (\sigma, \lambda)$	$(4\pi\epsilon_0)^{1/2}\rho, (\sigma, \lambda)$
Conductivity	σ	$4\pi\epsilon_0\sigma$
Current	I	$(4\pi\epsilon_0)^{1/2}I$
Current density	$\mathbf{J}, (\mathbf{K})$	$(4\pi\epsilon_0)^{1/2}\mathbf{J}, (\mathbf{K})$
Dielectric constant	κ_e	ϵ
Dipole moment (electric)	$\mathbf{p}$	$(4\pi\epsilon_0)^{1/2}\mathbf{p}$
Dipole moment (magnetic)	$\mathbf{m}$	$(4\pi/\mu_0)^{1/2}\mathbf{m}$
Displacement	$\mathbf{D}$	$(\epsilon_0/4\pi)^{1/2}\mathbf{D}$
Electric field	$\mathbf{E}$	$(4\pi\epsilon_0)^{-1/2}\mathbf{E}$
Inductance	L	$(4\pi\epsilon_0)^{-1}L$
Magnetic field	$\mathbf{H}$	$(4\pi\mu_0)^{-1/2}\mathbf{H}$
Magnetic flux	Φ	$(\mu_0/4\pi)^{1/2}\Phi$
Magnetic induction	$\mathbf{B}$	$(\mu_0/4\pi)^{1/2}\mathbf{B}$
Magnetization	$\mathbf{M}$	$(4\pi/\mu_0)^{1/2}\mathbf{M}$
Permeability	μ	(1) $\kappa_m\mu_0$, then (2) $\kappa_m \to \mu$
Permeability (relative)	κ_m	μ
Permittivity	ϵ	(1) $\kappa_e\epsilon_0$, then (2) $\kappa_e \to \epsilon$
Polarization	$\mathbf{P}$	$(4\pi\epsilon_0)^{1/2}\mathbf{P}$
Resistance	R	$(4\pi\epsilon_0)^{-1}R$
Resistivity	ρ	$(4\pi\epsilon_0)^{-1}\rho$
Scalar potential	ϕ	$(4\pi\epsilon_0)^{-1/2}\phi$
Speed of light	$(\mu_0\epsilon_0)^{-1/2}$	c
Susceptibility	$\chi_e, (\chi_m)$	$4\pi\chi_e, (\chi_m)$
Vector potential	$\mathbf{A}$	$(\mu_0/4\pi)^{1/2}\mathbf{A}$

■ **Example**

Let us transform $\nabla \cdot \mathbf{D} = \rho_f$ as given by (21-19). Using the table, we get $\nabla \cdot [(\epsilon_0/4\pi)^{1/2}\mathbf{D}] = (4\pi\epsilon_0)^{1/2}\rho_f$, which reduces to $\nabla \cdot \mathbf{D} = 4\pi\rho_f$ as quoted in (23-8). ■

■ **Example**

Let us transform the expression for the Poynting vector given in (23-14) into MKSA form. Since $\mathbf{S}$ is an energy flow, it is not changed and, by going from the Gaussian column to the MKSA one, we get

$$\mathbf{S} = \frac{(\mu_0\epsilon_0)^{-1/2}}{4\pi}\left[(4\pi\epsilon_0)^{1/2}\mathbf{E} \times (4\pi\mu_0)^{1/2}\mathbf{H}\right] = \mathbf{E} \times \mathbf{H}$$

which is exactly (21-59). ■

The use of Table 23-1 may occasionally lead to a wrong result when applied to a Gaussian system equation that has been worked out for a *vacuum*. The reason is that, since $\mathbf{D} = \mathbf{E}$ and $\mathbf{B} = \mathbf{H}$ in this case, there is a tendency to use these symbols interchangeably which may lead to ambiguities since the conversion factors listed in Table 23-1 are different for the members of each of these pairs. For example, in such a situation it is quite common to find the equation connecting the field with the vector potential written as $\mathbf{H} = \nabla \times \mathbf{A}$, and we quickly find from the table that this will not transform directly back to the corresponding MKSA equation $\mathbf{B} = \nabla \times \mathbf{A}$, although it does lead to $\mu_0 \mathbf{H} = \nabla \times \mathbf{A}$ which, *for a vacuum*, is all right.

The problem of converting the *form* of an equation is different from that of converting the *numerical values* of a given physical quantity from one unit system to another. For example, the data may be given numerically in Gaussian units and it is necessary to insert their equivalent values into an MKSA formula, or conversely. For this purpose, one requires a numerical conversion table and Table 23-2 is adequate for most purposes. The entry in each row gives the same amount of the given quantity expressed in different units; that is, the terms in each row are equal. The various factors of 3 arise from writing $c = 3 \times 10^8$ meters/second; this does not apply to powers of 10. Other needed conversions can be obtained in the usual manner by multiplying by unity as expressed by the appropriate ratio and canceling units; for example, one can write $1 = 10^3$ grams/1 kilogram.

Although **H** and **M** are both measured in ampere/meter in the MKSA system, we see that the conversion factors to oersted are different for each of them; this is a consequence of the 4π in (23-9), and similar remarks apply to **D** and **P**. It is not unusual to find magnetization stated in gauss rather than in oersted; in the overwhelming majority of such cases, the author *really* means "oersted" and one can proceed by

Table 23-2. Conversion table for numerical values

Quantity	MKSA	Gaussian
Length	1 meter (m)	10^2 centimeters (cm)
Mass	1 kilogram	10^3 grams
Time	1 second	1 second
Force	1 newton	10^5 dynes
Work, energy	1 joule	10^7 ergs
Power	1 watt	10^7 ergs/second
Capacitance (C)	1 farad	9×10^{11} statfarads
Charge (q)	1 coulomb	3×10^9 statcoulombs
Charge density (ρ)	1 coulomb/m^3	3×10^3 statcoulomb/cm^3
Conductivity (σ)	1 $(\text{ohm-m})^{-1}$	9×10^9 $(\text{statohm-cm})^{-1}$
Current (I)	1 ampere	3×10^9 statamperes $= 10^{-1}$ abamperes
Current density (**J**)	1 ampere/m^2	3×10^5 statampere/cm^2
Displacement (**D**)	1 coulomb/m^2	$12\pi \times 10^5$ statvolt/cm
Electric field (**E**)	1 volt/m	$\frac{1}{3} \times 10^{-4}$ statvolt/cm
Inductance (L)	1 henry	$\frac{1}{9} \times 10^{-11}$ stathenrys
Magnetic field (**H**)	1 ampere/m	$4\pi \times 10^{-3}$ oersted
Magnetic flux (Φ)	1 weber	10^8 maxwells
Magnetic induction (**B**)	1 weber/m^2 = 1 tesla	10^4 gauss
Magnetization (**M**)	1 ampere/m	10^{-3} oersted
Polarization (**P**)	1 coulomb/m^2	3×10^5 statvolt/cm
Potential (ϕ)	1 volt	$\frac{1}{300}$ statvolt
Resistance (R)	1 ohm	$\frac{1}{9} \times 10^{-11}$ statohms

changing the name and using the factor given in the table for **M**. Occasionally, the author *really* means "gauss"; this will normally signify that he or she has in mind an MKSA definition of magnetic dipole moment equal to μ_0 times the expression (19-20); this would make the relation among the magnetic vectors take the form $\mathbf{B} = \mu_0\mathbf{H} + \mathbf{M}$ rather than (21-24). In that case, it would be appropriate to measure **B** and **M** in the same units; this is a rare situation, however.

When one looks up numerical values of the quantities permeability, dielectric constant, and susceptibilities, one often finds them given in Gaussian terms. The numerical relations among these parameters in the two systems are:

$$\kappa_{e\,\mathrm{MKSA}} = \left(\frac{\epsilon}{\epsilon_0}\right)_{\mathrm{MKSA}} = \epsilon_{\mathrm{Gaussian}} \tag{23-16}$$

$$\kappa_{m\,\mathrm{MKSA}} = \left(\frac{\mu}{\mu_0}\right)_{\mathrm{MKSA}} = \mu_{\mathrm{Gaussian}} \tag{23-17}$$

$$\chi_{\mathrm{MKSA}} = 4\pi\chi_{\mathrm{Gaussian}} \tag{23-18}$$

where the last relation holds for both χ_e and χ_m. (Also see Exercise 20-17.)

EXERCISES

23-1 Using (23-6), express the dimensions of a statcoulomb in terms of centimeters, grams, and seconds. Similarly, use (23-7) to do the same for an abampere.

23-2 Show that 1 statcoulomb/(centimeter)^2 = 1 statvolt/centimeter. Also show that 1 statfarad = 1 centimeter, and that 1 statohm = 1 second/centimeter.

23-3 Show that all of the equations (23-8) through (23-13) can be obtained by applying Table 23-1 to the corresponding MKSA equations.

23-4 Beginning with the equations stated in Gaussian form, derive the differential equations satisfied by **A** and ϕ and the Lorentz condition for a l.i.h. medium. Verify that they are the same as those obtained with the use of Table 23-1.

23-5 Use (23-8) and (23-9) to obtain the capacitance of a parallel plate capacitor of plate area A and separation d with vacuum between the plates. Verify that your result is consistent with Table 23-1 and Exercise 23-2.

23-6 Use (23-8) to show that the induced emf will be written in Gaussian units as $\mathcal{E} = -c^{-1}(d\Phi/dt)$. If self-inductance is also defined in the usual way by $\mathcal{E} = -L(dI/dt)$, show that the analogue of (17-55) must be $L = \Phi/cI$. Then show that 1 Gaussian unit of inductance = 1 stathenry = 1 (second)^2/centimeter. Now use (23-8) and (23-9) to find the self-inductance of a length l of an infinitely long ideal solenoid of cross-sectional area S, n turns per unit length, and vacuum inside. Verify that your result is consistent with Table 23-1 and the above result for its dimensions.

23-7 Use (23-8) and (23-11) to derive Poynting's theorem for a linear isotropic medium and thus show the suitability of the results quoted in (23-14).

23-8 As a simple numerical exercise in the use of Table 23-2, suppose that **H** and **M** are parallel and that they have the values α ampere/meter and β ampere/meter, respectively, where α and β are numbers. Find B in webers/(meter)^2, B in gauss, H in oersted, and M in oersted. Show that when the values just found for H and M are put into (23-9), the same value of B is obtained as found by direct use of the conversion factor for B itself.

23-9 Verify that (23-16) through (23-18) are correct. (*Hint*: as in the previous exercise, choose a specific numerical value for the appropriate quantity and carry out all of the conversions.)