APPENDIX E

THE LAGRANGIAN FOR A CHARGE q IN A MAGNETIC FIELD

How do we handle magnetic fields within the framework of a Lagrangian? Purely electric forces are easy. After all, the electric potential $\varphi(\mathbf{r})$ is introduced in electrostatics as the work done per unit charge to bring the charge to the position $\mathbf{r}$ from some reference point, which is often taken as at infinity. Then the potential energy of a charge q is $V = q\varphi$ and the Lagrangian is given by

$$L = T - V = \tfrac{1}{2}m\mathbf{v}^2 - q\varphi \tag{E.1}$$

In terms of the Cartesian coordinates $x_1 = x$, $x_2 = y$, and $x_3 = z$, the Euler-Lagrange equation of motion

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0 \tag{E.2}$$

for the Lagrangian (E.1) is given by

$$-q\frac{\partial \varphi}{\partial x_i} - \frac{d}{dt}m\dot{x}_i = 0 \tag{E.3}$$

This equation of motion is simply

$$m\ddot{x}_i = -q\frac{\partial \varphi}{\partial x_i} \tag{E.4}$$

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$$-q\frac{\partial \varphi}{\partial x_i} + q$$

Since the electric field $\mathbf{E}$ is given in electrostatics by $\mathbf{E} = -\nabla\varphi$, the equation of motion can be expressed in terms of vectors as the force law $m\mathbf{a} = \mathbf{F} = q\mathbf{E}$.

The full Lorentz force

$$\mathbf{F} = q\mathbf{E} + q(\mathbf{v}/c) \times \mathbf{B} \tag{E.5}$$

includes velocity-dependent magnetic forces, which cannot be obtained from a Lagrangian of the form (E.1) that is just the difference of the kinetic and potential energies. Since the magnetic force always acts at right angles to the velocity, it doesn't change the magnitude of the velocity and thus does no work. However, we can show that the Lagrangian

$$L = \frac{1}{2}m\mathbf{v}^2 - q\varphi + \frac{q}{c}\mathbf{A}\cdot\mathbf{v} \tag{E.6}$$

which differs from (E.1) by the addition of a velocity-dependent term involving the vector potential $\mathbf{A}$, yields the Lorentz force (E.5) for the equations of motion. The magnetic field $\mathbf{B}$ can always be expressed in the form $\mathbf{B} = \nabla \times \mathbf{A}$, since the magnetic field satisfies $\nabla\cdot\mathbf{B} = 0$ and the gradient of a curl vanishes:

$$\nabla\cdot\mathbf{B} = \nabla\cdot(\nabla\times\mathbf{A}) = 0 \tag{E.7}$$

Since for the Lagrangian (E.6)

$$\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + \frac{q}{c}A_i \tag{E.8}$$

the canonical momentum $p_i = \partial L/\partial \dot{x}_i$ is given in vector form by

$$\mathbf{p} = m\mathbf{v} + \frac{q}{c}\mathbf{A} \tag{E.9}$$

In order to evaluate

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = m\ddot{x}_i + \frac{q}{c}\frac{dA_i}{dt} \tag{E.10}$$

notice that $A_i = A_i[x(t), y(t), z(t), t]$ and therefore

$$\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + \sum_{j=1}^{3}\frac{\partial A_i}{\partial x_j}\frac{dx_j}{dt} = \frac{\partial A_i}{\partial t} + \mathbf{v}\cdot\nabla A_i \tag{E.11}$$

Using

$$\frac{\partial L}{\partial x_i} = -q\frac{\partial\varphi}{\partial x_i} + \frac{q}{c}\mathbf{v}\cdot\frac{\partial \mathbf{A}}{\partial x_i} \tag{E.12}$$

(E.2) becomes

$$-q\frac{\partial\varphi}{\partial x_i} + q\frac{\mathbf{v}}{c}\cdot\frac{\partial\mathbf{A}}{\partial x_i} - m\ddot{x}_i - \frac{q}{c}\left(\frac{\partial A_i}{\partial t} + \mathbf{v}\cdot\nabla A_i\right) = 0 \tag{E.13}$$

or

$$m\ddot{x}_i = -q\frac{\partial\varphi}{\partial x_i} + q\frac{\mathbf{v}}{c}\cdot\frac{\partial\mathbf{A}}{\partial x_i} - \frac{q}{c}\left(\frac{\partial A_i}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}A_i\right) \tag{E.14}$$

In vector notation, (E.14) can be expressed in terms of the force **F** on the particle as

$$\mathbf{F} = q\left(-\boldsymbol{\nabla}\varphi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) + \frac{q}{c}\left[\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}\right] \tag{E.15}$$

or

$$\mathbf{F} = q\mathbf{E} + \frac{q}{c}\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A}) = q\mathbf{E} + \frac{q}{c}\mathbf{v}\times\mathbf{B} \tag{E.16}$$

as desired.

Given the Lagrangian (E.6), we can determine the Hamiltonian in the usual way:

$$\begin{aligned} H &= \sum_{i=1}^{3} p_i\dot{x}_i - L \\ &= \sum_{i=1}^{3}\left(m\dot{x}_i + \frac{q}{c}A_i\right)\dot{x}_i - \left(\sum_{i=1}^{3}\frac{1}{2}m\dot{x}_i\dot{x}_i - q\varphi + \frac{q}{c}\sum_{i=1}^{3}A_i\dot{x}_i\right) \\ &= \sum_{i=1}^{3}\frac{1}{2}m\dot{x}_i\dot{x}_i + q\varphi \end{aligned} \tag{E.17}$$

At first it appears that the vector potential has disappeared entirely from the Hamiltonian. However, if we express the Hamiltonian in terms of the canonical momentum (E.9), we obtain

$$H = \frac{(\mathbf{p} - q\mathbf{A}/c)^2}{2m} + q\varphi \tag{E.18}$$

This suggests a mnemonic for the way to turn on electromagnetic interactions in terms of the Hamiltonian: take the energy for a free particle of charge q

$$E = \frac{\mathbf{p}^2}{2m} \tag{E.19}$$

and make the replacements $\mathbf{p} \to \mathbf{p} - q\mathbf{A}/c$ and $E \to E - q\varphi$ to generate (E.18), with the energy E replaced by the symbol for the Hamiltonian.

1 Å ≡ 10^{-10} m
1 fm ≡ 10^{-15} m
1 barn ≡ 10^{-28} m^2
1 dyne ≡ 10^{-5} newton (N
1 gauss(G) ≡ 10^{-4} tesla (T
1 erg ≡ 10^{-7} joule (J)

Quantity	Symbo
Speed of light	c
Planck's constant	h
	$\hbar = h$
Electron charge	e
Electron mass	m_e