Quantum Mechanics 2 Homework #9

1) Goswami problem 16.3. Plug in $a = D \cdot \xi$, $b = -A$, and $c = E_0$ (as those are defined in Section 16.2) and interpret the results.

2) Goswami problem 16.8.

3) In an NMR experiment, a sample of water displays resonant absorption when the frequency of the transverse magnetic field has a value of 42.3 MHz. If the parallel magnetic field B_0 is 10000 gauss, what is the value of g for the proton?

4) Goswami problem 18.2

5) Goswami problem 18.4.

QM2

ANSWERS TO HOMEWORK 9

(1) $H = a\sigma_z + b\sigma_x + cI$

$$= a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+c & b \\ b & -a+c \end{pmatrix}$$

Diagonalize:

$$\begin{vmatrix} a+c-\lambda & b \\ b & -a+c-\lambda \end{vmatrix} = 0$$

$$(a+c-\lambda)(-a+c-\lambda) - b^2 = 0$$

$$\lambda_\pm = c \pm \sqrt{a^2+b^2}$$

The eigenvectors are

For λ_+: $$\begin{pmatrix} u_+ \\ v_+ \end{pmatrix} = \frac{1}{\sqrt{2}\,\sqrt[4]{a^2+b^2}}\begin{pmatrix} \left[a+\sqrt{a^2+b^2}\right]^{1/2} \\ \left[-a+\sqrt{a^2+b^2}\right]^{1/2}\cdot b/|b| \end{pmatrix} = \text{"}|\alpha\rangle\text{"}$$

For λ_-: $$\begin{pmatrix} u_- \\ v_- \end{pmatrix} = \frac{1}{\sqrt{2}\,\sqrt[4]{a^2+b^2}}\begin{pmatrix} -\left[-a+\sqrt{a^2+b^2}\right]^{1/2}\cdot b/|b| \\ \left[a+\sqrt{a^2+b^2}\right]^{1/2} \end{pmatrix} = \text{"}|\beta\rangle\text{"}$$

Plug in $a = D\mathcal{E}$

$b = -A$

$c = E_0$

If $A > 0$,

$$\boxed{\begin{aligned} |\alpha\rangle &= \frac{1}{\sqrt{2}\left[(D\mathcal{E})^2+A^2\right]^{1/4}}\begin{pmatrix} \left[\sqrt{(D\mathcal{E})^2+A^2}+D\mathcal{E}\right]^{1/2} \\ -\left[\sqrt{(D\mathcal{E})^2+A^2}-D\mathcal{E}\right]^{1/2} \end{pmatrix} \\ |\beta\rangle &= \frac{1}{\sqrt{2}\left[(D\mathcal{E})^2+A^2\right]^{1/4}}\begin{pmatrix} \left[\sqrt{(D\mathcal{E})^2+A^2}-D\mathcal{E}\right]^{1/2} \\ \left[\sqrt{(D\mathcal{E})^2+A^2}+D\mathcal{E}\right]^{1/2} \end{pmatrix} \end{aligned}}$$

2 From Gosnami Eq 16.41,

(a) $H = -\mu \vec{\sigma} \cdot \vec{B} = -\frac{\mu B_0}{\sqrt{2}}(\sigma_y + \sigma_z) = \frac{-\mu B_0}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$

(b) Diagonalize, the eigenvalues are

$\lambda_\pm = \pm \mu B_0$

The eigenvectors are

$$\begin{pmatrix} u_+ \\ v_+ \end{pmatrix} = \frac{1}{\sqrt{2\sqrt{2}}} \begin{pmatrix} [\sqrt{2}-1]^{1/2} \\ -i[\sqrt{2}+1]^{1/2} \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} u_- \\ v_- \end{pmatrix} = \frac{1}{\sqrt{2\sqrt{2}}} \begin{pmatrix} [\sqrt{2}+1]^{1/2} \\ i[\sqrt{2}-1]^{1/2} \end{pmatrix}$$

(c) The probability of measuring $+\mu B_0$ is

$$\left| \frac{1}{\sqrt{2\sqrt{2}}} \left([\sqrt{2}-1]^{1/2}, +i[\sqrt{2}+1]^{1/2} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right|^2 = \boxed{\frac{1}{2\sqrt{2}}[\sqrt{2}+1]}$$

The probability of measuring $-\mu B_0$ is

$$\left| \frac{1}{\sqrt{2\sqrt{2}}} \left([\sqrt{2}+1]^{1/2}, -i[\sqrt{2}-1]^{1/2} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right|^2 = \boxed{\frac{1}{2\sqrt{2}}[\sqrt{2}-1]}$$

(d) $$|\psi(t)\rangle = a e^{-i\mu B_0 t/\hbar} \frac{1}{\sqrt{2\sqrt{2}}} \left\{ [\sqrt{2}-1]^{1/2}|+\rangle - i[\sqrt{2}+1]^{1/2}|-\rangle \right\}$$

$$+ b e^{+i\mu B_0 t/\hbar} \frac{1}{\sqrt{2\sqrt{2}}} \left\{ [\sqrt{2}+1]^{1/2}|+\rangle + i[\sqrt{2}-1]^{1/2}|-\rangle \right\}$$

To find "a" and "b", notice that $|\psi(t=0)\rangle = |-\rangle$, so

$$a[\sqrt{2}-1]^{1/2} + b[\sqrt{2}+1]^{1/2} = 0 \quad \text{and}$$
$$a[\sqrt{2}+1]^{1/2} - b[\sqrt{2}-1]^{1/2} = i\sqrt{2\sqrt{2}}$$

Solve simultaneously:

$$a = \frac{i[\sqrt{2}+1]^{1/2}}{\sqrt{2\sqrt{2}}} \quad \text{and}$$

$$b = \frac{-i[\sqrt{2}-1]^{1/2}}{\sqrt{2\sqrt{2}}}$$

Then
$$\boxed{|\Psi(t)\rangle = \frac{1}{2\sqrt{2}}\left[\exp\left(\frac{-i\mu B_0 t}{\hbar}\right)\left\{i|+\rangle + (\sqrt{2}+1)|-\rangle\right\} + \exp\left(\frac{+i\mu B_0 t}{\hbar}\right)\left\{-i|+\rangle + (\sqrt{2}-1)|-\rangle\right\}\right]}$$

$$\langle\Psi(t)|S_y|\Psi(t)\rangle = \frac{\hbar}{2}\langle\Psi(t)|\sigma_y|\Psi(t)\rangle$$

$$= \boxed{-\frac{\hbar}{4}(1-\cos\omega t)}$$

3. The condition for resonance was seen to be

$$\omega = \frac{2\mu_p B_0}{\hbar} \quad \text{where}$$

$$\mu_p \equiv \frac{-e g_p \hbar}{4mc}$$

So $\omega = \frac{2B_0}{\cancel{\hbar}}\left(\frac{e g_p \cancel{\hbar}}{4mc}\right)$

$$g_p = \frac{2mc\omega}{B_0 e}$$

$$= \frac{2(1.67\times10^{-27}\,kg)(42.3\times10^{6}\,Hz)}{\left(10000\text{ gauss}\times\frac{1\text{ tesla}}{10^4\text{ gauss}}\right)(1.6\times10^{-19}C)} = 0.88$$

4. $H_1 = Cx^3$. Use Gasiorowicz Equation 15.17 for energy corrections:

$$\langle \varphi_k | Cx^3 | \varphi_1 \rangle = \begin{cases} C(\hbar/2m\omega)^{3/2}\sqrt{24} & \text{for } k=4 \\ C(\hbar/2m\omega)^{3/2}\,6\sqrt{2} & \text{for } k=2 \\ C(\hbar/2m\omega)^{3/2}\cdot 3 & \text{for } k=0 \\ 0 & \text{otherwise} \end{cases}$$

Recall $E_1^{(2)} = \sum_{k\neq 1} \frac{|\langle \varphi_k | H_1 | \varphi_1 \rangle|^2}{E_1^{(0)} - E_k^{(0)}}$. Plug the above into this:

$$E_1^{(2)} = \frac{\left[C(\hbar/2m\omega)^{3/2}\sqrt{24}\right]^2}{\left[(\tfrac{1}{2}+1)\hbar\omega - (\tfrac{1}{2}+4)\hbar\omega\right]} + \frac{\left[C(\hbar/2m\omega)^{3/2}\,6\sqrt{2}\right]^2}{\left[(\tfrac{1}{2}+1)\hbar\omega - (\tfrac{1}{2}+2)\hbar\omega\right]}$$

$$+ \frac{\left[C(\hbar/2m\omega)^{3/2}\cdot 3\right]^2}{\left[(\tfrac{1}{2}+1)\hbar\omega - \tfrac{1}{2}\hbar\omega\right]}$$

$$\boxed{= -\frac{71 C^2 \hbar^2}{8 m^3 \omega^4}}.$$

To find $E_2^{(2)} = \sum_{k\neq 2} \frac{|\langle \varphi_k | H_1 | \varphi_2 \rangle|^2}{E_2^{(0)} - E_k^{(0)}}$, use

$$\langle \varphi_k | Cx^3 | \varphi_2 \rangle = \begin{cases} C(\hbar/2m\omega)^{3/2}\sqrt{60} & \text{for } k=5 \\ C(\hbar/2m\omega)^{3/2}\,9\sqrt{3} & \text{for } k=3 \\ C(\hbar/2m\omega)^{3/2}\,6\sqrt{2} & \text{for } k=1 \\ 0 & \text{otherwise} \end{cases}$$

Then $E_2^{(2)} = \frac{C^2(\hbar/2m\omega)^3}{\hbar\omega}\left\{\frac{60}{2-5} + \frac{243}{2-3} + \frac{72}{2-1}\right\} = \boxed{\frac{-191 C^2\hbar^2}{8m^3\omega^4}}$

Use Goswami Eq 18.12 for wavefunction corrections:

$$|\psi_n\rangle = |\varphi_n\rangle + \sum_{k\neq n} \frac{\langle\varphi_k|H_1|\varphi_n\rangle}{E_n^{(0)}-E_k^{(0)}}|\varphi_k\rangle$$

$$\text{So } |\psi_1^{(1)}\rangle = \frac{C(\hbar/2m\omega)^{3/2}\sqrt{2}\;|\varphi_4\rangle}{\left[(\frac{1}{2}+1)\hbar\omega-(\frac{1}{2}+4)\hbar\omega\right]} + \frac{C(\hbar/2m\omega)^{3/2}6\sqrt{2}\;|\varphi_2\rangle}{\left[(\frac{1}{2}+1)\hbar\omega-(\frac{1}{2}+2)\hbar\omega\right]}$$

$$+\frac{C(\hbar/2m\omega)^{3/2}3\;|\varphi_0\rangle}{\left[(\frac{1}{2}+1)\hbar\omega-\frac{1}{2}\hbar\omega\right]}$$

$$\text{So } \boxed{\langle x|\psi_1^{(1)}\rangle = \frac{C(\hbar/2m\omega)^{3/2}}{\hbar\omega}\left\{-\frac{\sqrt{24}}{3}u_4(x)-6\sqrt{2}\,u_2(x)+3u_0(x)\right\}}$$

where the u_i are normalized simple harmonic oscillator eigenfunctions given by Goswami Eq. 7.18

Similarly,

$$\boxed{\langle x|\psi_2^{(2)}\rangle = \frac{C(\hbar/2m\omega)^{3/2}}{\hbar\omega}\left\{-\frac{\sqrt{60}}{3}u_5(x)-9\sqrt{3}\,u_3(x)+6\sqrt{2}\,u_1(x)\right\}}$$

5. $H_1 = Cx^4$

$$\langle \varphi_{n'} | Cx^4 | \varphi_n \rangle = C \sum_{m,k,\ell} \langle \varphi_{n'} | x | \varphi_m \rangle \langle \varphi_m | x | \varphi_k \rangle \langle \varphi_k | x | \varphi_\ell \rangle \langle \varphi_\ell | x | \varphi_n \rangle$$

$$= C(\hbar/2m\omega)^2 \cdot \begin{cases} [(n+1)(n+2)(n+3)(n+4)]^{1/2} & \text{if } n' = n+4 \\ [n(n-1)(n-2)(n-3)]^{1/2} & \text{if } n' = n-4 \\ (4n+6)[(n+1)(n+2)]^{1/2} & \text{if } n' = n+2 \\ (4n-2)[n(n-1)]^{1/2} & \text{if } n' = n-2 \\ (n+1)(n+2) + n(n-1) + n^2 + (n+1)^2 + 2n(n+1) & \text{if } n' = n \\ 0 & \text{otherwise} \end{cases}$$

(i) 1st state: $n = 0$

$$\langle \varphi_{n'} | Cx^4 | \varphi_0 \rangle = C(\hbar/2m\omega)^2 \cdot \begin{cases} \sqrt{24} & \text{if } n' = 4 \\ 6\sqrt{2} & \text{if } n' = 2 \\ 3 & \text{if } n' = 0 \\ 0 & \text{otherwise} \end{cases}$$

From Goswami Eq. 18.10: $E_0^{(1)} = \langle \varphi_0 | Cx^4 | \varphi_0 \rangle = 3C(\hbar/2m\omega)^2$

From Goswami Eq 18:14:

$$E_0^{(2)} = \frac{C^2(\hbar/2m\omega)^4}{\hbar\omega} \left\{ \frac{24}{0-4} + \frac{72}{0-2} \right\} = \frac{-21 C^2 \hbar^3}{8 m^4 \omega^5}$$

(ii) 2ND State: $n = 1$

Similar calculations lead to

$$E_1^{(1)} = 15C(\hbar/2m\omega)^2$$

$$E_1^{(2)} = \frac{-165 C^2 \hbar^3}{8 m^4 \omega^5}$$