Quantum Mechanics 2 Homework #8

1) Goswami problem 15.12.

2) Recall that any two linearly independent spinors span the space of complex two-dimensional vectors.

(a) Write down the eigenspinors S_x, S_y, and S_z in the S_z basis. This amounts to 6 eigenspinors.

(b) Show that any generic spinor, $\begin{pmatrix} a \\ b \end{pmatrix}$, may be expressed as a linear combination of any one of these three pairs of eigenspinors.

3) Consider the matrix $A = \begin{pmatrix} \gamma & 0 & i\beta\gamma \\ 0 & 1 & 0 \\ -i\beta\gamma & 0 & \gamma \end{pmatrix}$. It can be diagonalize by a matrix U.

(a) Find U.

(b) Show that the eigenvectors comprising U are orthogonal.

(c) Carry out the matrix multiplication to verify that $U^{-1}AU$ is diagonal.

(d) Show that U is unitary by obtaining $U^\dagger$ and verifying by matrix multiplication that $UU^\dagger = 1$.

4) Consider an angular momentum = 1 system represented by a state vector

$\psi = \frac{1}{\sqrt{26}}\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$. What is the probability that a measurement of L_x yields the value 0?

5) Construct a 2×2 unitary matrix that has at least 2 imaginary elements.

Answers to homework 8

1. $M^{+} = (M^{T})^{*} = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & i \\ 0 & -i & 0 \end{pmatrix}^{*} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = M.$

So M is Hermitian.

To find the eigenvalues:

$$0 = \begin{vmatrix} -\lambda & -i & 0 \\ i & -\lambda & -i \\ 0 & i & -\lambda \end{vmatrix} = -\lambda^3 + 2\lambda$$

So $\boxed{\begin{array}{l} \lambda_1 = \sqrt{2} \\ \lambda_2 = 0 \\ \lambda_3 = -\sqrt{2} \end{array}}$

To find eigenvector #1,

$$\begin{pmatrix} -\sqrt{2} & -i & 0 \\ i & -\sqrt{2} & -i \\ 0 & i & -\sqrt{2} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \boxed{\begin{pmatrix} \frac{1}{2} \\ i/\sqrt{2} \\ -1/2 \end{pmatrix} = \text{eigenvector } 1}$$

Similarly,

$$\text{eigenvector } 2 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$\text{eigenvector } 3 = \begin{pmatrix} \frac{1}{2} \\ -i/\sqrt{2} \\ \end{pmatrix}$$

U can be constructed from these eigenvectors as explained on page 305:

$$\boxed{U = \begin{pmatrix} \frac{1}{2} & \frac{-1}{\sqrt{2}} & \frac{-1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{i}{\sqrt{2}} & \frac{-1}{2} \end{pmatrix}}$$

$$U^\dagger = (U^T)^* = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{-i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ \frac{-1}{2} & \frac{1}{\sqrt{2}} & \frac{-1}{2} \end{pmatrix}^* = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ \frac{-1}{2} & \frac{1}{\sqrt{2}} & \frac{-1}{2} \end{pmatrix}$$

$U^{-1} = \frac{1}{\det U}\tilde{C}$, where C are the cofactors

$\det U = \frac{-i}{4}\,\frac{-i}{4}\,\frac{-i}{4}\,\frac{-i}{4} = \boxed{-i}$, so $\frac{1}{\det U} = i$

$$C = \begin{pmatrix} \frac{-i}{2} & (-1)\cdot\left(\frac{-i}{\sqrt{2}}\right) & \frac{+i}{2} \\ (-1)\cdot\frac{i}{\sqrt{2}} & 0 & (-1)\cdot\frac{i}{\sqrt{2}} \\ \frac{-i}{2} & (-1)\cdot\left(\frac{i}{\sqrt{2}}\right) & \frac{+i}{2} \end{pmatrix} = \begin{pmatrix} \frac{-i}{2} & \frac{1}{\sqrt{2}} & \frac{i}{2} \\ \frac{-i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ \frac{-i}{2} & \frac{-1}{\sqrt{2}} & \frac{i}{2} \end{pmatrix}$$

So $\tilde{C} = \begin{pmatrix} \frac{-i}{2} & \frac{-i}{\sqrt{2}} & \frac{-i}{2} \\ \frac{i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ \frac{i}{2} & \frac{-i}{\sqrt{2}} & \frac{i}{2} \end{pmatrix}$

So $U^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ \frac{-1}{2} & \frac{1}{\sqrt{2}} & \frac{-1}{2} \end{pmatrix}$

So U is not unitary

② (a) $\chi_+^x = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, $\chi_-^x = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$\chi_+^y = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$, $\chi_-^y = \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$

$\chi_+^z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_-^z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}\left[(a+b)\chi_+^x + (a-b)\chi_-^x\right]$

$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}\left[(a-ib)\chi_+^y + (a+ib)\chi_-^y\right]$

$\begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+^z + b\chi_-^z$

3. (a) Using the "recipe" given in class and on pp. 305–306,

$$U = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ i/\sqrt{2} & 0 & i/\sqrt{2} \end{pmatrix}$$

(b) The eigenvectors are

$$u_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -i/\sqrt{2} \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ -i/\sqrt{2} \end{pmatrix}$$

By direct demonstration, $u_i u_j = \delta_{ij}$

(c) Find U^{-1}:

$$\det U = \frac{i}{2} + \frac{i}{2} = i$$

So $\frac{1}{\det U} = -i$

$$C = \begin{pmatrix} i/\sqrt{2} & (-1)\cdot 0 & -i/\sqrt{2} \\ (-1)\cdot 0 & i & (-1)\cdot 0 \\ 1/\sqrt{2} & (-1)\cdot 0 & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & i & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$\tilde{C} = \begin{pmatrix} i/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & i & 0 \\ -i/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

So $U^{-1} = \frac{\tilde{C}}{\det U} = \begin{pmatrix} +1/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & +1 & 0 \\ -1/\sqrt{2} & 0 & -i/\sqrt{2} \end{pmatrix}$

$U^{-1} A U =$

$$\begin{pmatrix} 1/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & -i/\sqrt{2} \end{pmatrix} \begin{pmatrix} \gamma & 0 & i\beta\gamma \\ 0 & 1 & 0 \\ -i\beta\gamma & 0 & \gamma \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ i/\sqrt{2} & 0 & i/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma(1-\beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma(1+\beta) \end{pmatrix}$$

(d) $U^{\dagger} = \begin{pmatrix} 1/\sqrt{2} & 0 & i/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & i/\sqrt{2} \end{pmatrix}^{*}$

$$= \begin{pmatrix} 1/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & -i/\sqrt{2} \end{pmatrix}$$

$$U U^{\dagger} = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ i/\sqrt{2} & 0 & i/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & -i/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. When $\ell=1$, $L_x = \begin{pmatrix} 0 & \hbar\sqrt{2} & 0 \\ \hbar\sqrt{2} & 0 & \hbar\sqrt{2} \\ 0 & \hbar\sqrt{2} & 0 \end{pmatrix}$

This has 3 eigenvalues. The eigenvector associated with eigenvalue $=0$ is

$$\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$$

So $\left| \langle f | \alpha \rangle \langle \alpha | i \rangle \right|^2 =$

$$\left| \left(1/\sqrt{2},\ 0,\ -1/\sqrt{2} \right) \begin{pmatrix} 1/\sqrt{26} \\ 4/\sqrt{26} \\ -3/\sqrt{26} \end{pmatrix} \right|^2 = \boxed{\frac{4}{13}}$$

5. Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

If X is unitary, $X^{+} = X^{-1}$

This means $\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ ("Eq 1")

Assume $a^* = a$ (a is real)
$b^* = -b$ (b is imaginary)
$c^* = -c$ (c is imaginary)
$d^* = d$ (d is real)

The matrix Eq 1 can be written as 4 ordinary equations:

$$a = \frac{d}{ad-bc}$$

$$-b = \frac{-c}{ad-bc}$$

$$-c = \frac{-b}{ad-bc}$$

$$d = \frac{a}{ad-bc}$$

Choose: $a = d$
$b = c$
$ad - bc = 1$

Then with proper normalization,

$$\boxed{X = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}$$