Quantum Mechanics 2 Homework #7

1) Derive the eigenspinors of S_y.

2) Goswami problem 15.4.

3) Goswami problem 15.8.

4) Goswami problem 15.9.

5) Goswami problem 15.10.

QM 2

Answers to homework 7

(1) $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}$

$$\begin{vmatrix} -\lambda & -i\hbar/2 \\ i\hbar/2 & -\lambda \end{vmatrix} = 0$$

$$\downarrow$$

$$\lambda^2 - \frac{\hbar^2}{4} = 0$$

$$\downarrow$$

$$\lambda = \pm \frac{\hbar}{2}$$

To find χ_+^y:

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}\begin{pmatrix} u_+^y \\ v_+^y \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} u_+^y \\ v_+^y \end{pmatrix}$$

$$\downarrow$$

$$-iv_+^y = u_+^y$$

$$|u_+^y|^2 + |v_+^y|^2 = 1$$

So $\boxed{\chi_+^y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}}$

To find χ_-^y:

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}\begin{pmatrix} u_-^y \\ v_-^y \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} u_-^y \\ v_-^y \end{pmatrix}$$

$$\downarrow$$

$$-iv_-^y = -u_-^y$$

$$\downarrow$$

$|u_-^y|^2 + |v_-^y|^2 = 1$, so $\boxed{\chi_-^y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}}$

2. Consider an arbitrary matrix $A \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

a. Given: $[\sigma_z, A] = 0$

$$\downarrow$$

$$\sigma_z A - A\sigma_z = 0$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0$$

$$\downarrow$$

$$\begin{pmatrix} a & b \\ -c & -d \end{pmatrix} - \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} = 0$$

$$\downarrow$$

$$\begin{pmatrix} 0 & 2b \\ -2c & 0 \end{pmatrix} = 0$$

$\therefore\ b = c = 0$

Given $[\sigma_x, A] = 0$

$$\downarrow$$

$$\sigma_x A - A\sigma_x = 0$$

$$\downarrow$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0$$

$$\downarrow$$

$$\begin{pmatrix} 0 & d \\ a & 0 \end{pmatrix} - \begin{pmatrix} 0 & a \\ d & 0 \end{pmatrix} = 0$$

$$\downarrow$$

$$\begin{pmatrix} 0 & (d-a) \\ (a-d) & 0 \end{pmatrix} = 0 \longrightarrow a = d$$

So $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) Given $\{\sigma_z, A\} = 0$

$$\downarrow$$

$$\sigma_z A + A\sigma_z = 0$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0$$

$$\downarrow$$

$$\begin{pmatrix} 2a & 0 \\ 0 & -2d \end{pmatrix} = 0$$

$$\downarrow$$

$$a = d = 0$$

Given $\{\sigma_x, A\} = 0$

$$\downarrow$$

$$\sigma_x A + A\sigma_x = 0$$

$$\downarrow$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0$$

$$\downarrow$$

$$\begin{pmatrix} (c+b) & 0 \\ 0 & (b+b) \end{pmatrix} = 0$$

$$\downarrow$$

$$b = -c$$

$\therefore A = b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = bi\sigma_y$

So $\{\sigma_y, A\} = \sigma_y A + A\sigma_y$

$= \sigma_y (bi\sigma_y) + (bi\sigma_y)\sigma_y$

$= 2bi$ (From Goswami Eq. 15.296)

$\neq 0$ if $b \neq 0$

So A does not anticommute with σ_y if it anticommutes with σ_z and σ_z, unless $A = 0$.

③ From the last sentence on page 305, we know we need ⑤
the eigenspinors of S_x in the S_z basis. These are given in
Eq. 15.16 and 15.17

Then $U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

$$US_xU^{-1} = \frac{\hbar}{2}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} \hbar/2 & 0 \\ 0 & \hbar/2 \end{pmatrix}}$$

④ The transformation matrix from the S_z-basis to the S_x-basis is given by U in the problem above. So the representation of the given spinor in the S_x basis is

$$U\begin{pmatrix} 1/\sqrt{3} \\ \sqrt{\frac{2}{3}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix} = \boxed{\begin{pmatrix} \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \end{pmatrix}}$$

5 Equation 15.32 gives the transformation matrix U from the S_z basis to the S_y basis:

$$U = \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix}$$

To convert each of the χ_i to the S_y basis, multiply them by U:

S_z-basis	S_y-basis
$\chi_1 = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$
$\chi_2 = \begin{pmatrix} -i/\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{-i}{\sqrt{3}} \\ \sqrt{2}/\sqrt{3} \end{pmatrix} = \boxed{\begin{pmatrix} \frac{-i}{\sqrt{6}} - \frac{i}{\sqrt{3}} \\ \frac{-i}{\sqrt{6}} + \frac{i}{\sqrt{3}} \end{pmatrix}}$

$$\chi_1^\dagger \chi_2 = \left(1/\sqrt{2}, \ -i/\sqrt{2} \right)\begin{pmatrix} -i/\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{pmatrix} = \frac{-i}{\sqrt{6}} - \frac{i}{\sqrt{3}} \quad \text{in the } S_z\text{-basis}$$

$$\chi_1^\dagger \chi_2 = (1, \ 0)\begin{pmatrix} \frac{-i}{\sqrt{6}} - \frac{i}{\sqrt{3}} \\ \frac{-i}{\sqrt{6}} + \frac{i}{\sqrt{3}} \end{pmatrix} = \frac{-i}{\sqrt{6}} - \frac{i}{\sqrt{3}} \quad \text{in the } S_y\text{-basis}$$