Quantum Mechanics 2 Homework #6

1) Find the equations of motion of the spin operators $S_x(t)$, $S_y(t)$, and $S_z(t)$ in the presence of a Hamiltonian given by $H = eg\vec{S}(t)\cdot\vec{B}/2\mu c$. ($\vec{B}$ is magnetic field, e is electric charge, μ is particle mass, c is the speed of light, and g is a unitless constant.) Use the fact that $\left[S_x(t), S_y(t)\right] = i\hbar S_z(t)$ and cyclically. Consider the case in which $\vec{B} = (0,0,B)$ and solve for $\vec{S}(t)$ in terms of $\vec{S}(t=0)$.

2) Goswami problem 15.2.

3) Use the angular momentum ladder operators, customized for spin, to find the matrix representations of the operators S_x, S_y, and S_z for a spin-3/2 particle.

4) Goswami problem 15.A10. Do only the first of the three operators given in part (b).

5) Consider the Hamiltonian for the one-dimensional infinite square well. The bottom of the well lies between $x = 0$ and $x = L$. The goal of this problem is to write the matrix representation for this Hamiltonian.

(a) Write down the eigenfunctions of this Hamiltonian.

(b) Are these eigenfunctions a valid basis set for representing this Hamiltonian? Explain why or why not.

(c) If your answer to part (b) is "no," write down a basis that is appropriate for this Hamiltonian.

(d) Arrange the basis functions that you have chosen (from part (b) or part (c)) as row and column headings of a matrix. Find the Hamiltonian matrix in terms of the basis. If the matrix does not come out diagonal, diagonalize it and indicate the basis in which it is diagonal. A basis in which a Hamiltonian is diagonal constitutes "the energy representation" (in contrast to the x-space or p-space representation.)

Answers to homework 6

1. From Goswami Eq. 8.5,

$$\frac{dS_i(t)}{dt} = \frac{i}{\hbar}[H, S_i]$$

Here $H = \frac{eg}{2\mu c} \vec{S} \cdot \vec{B}$

So $$\frac{d}{dt}S_x = \frac{i}{\hbar}\left[\frac{eg}{2\mu c}\vec{S}\cdot\vec{B}, S_x\right]$$

$$= \frac{ieg}{2\hbar\mu c}[S_x B_x + S_y B_y + S_z B_z, S_x]$$

$$= \frac{ieg}{2\hbar\mu c}\Big\{S_x \underbrace{[B_x, S_x]}_{0} + \underbrace{[S_x, S_x]}_{0} B_x + S_y \underbrace{[B_y, S_x]}_{0} +$$

$$\underbrace{[S_y, S_x]}_{-i\hbar S_z} B_y + S_z \underbrace{[B_z, S_x]}_{0} + \underbrace{[S_z, S_x]}_{+i\hbar S_y} B_z\Big\}$$

$$= \frac{ieg}{2\hbar\mu c} i\hbar (S_y B_z - S_z B_y)$$

$$= \frac{eg}{2\mu c}(S_z B_y - B_z S_y) \qquad \text{"Eq 1"}$$

Similarly,

$$\frac{dS_y}{dt} = \frac{eg}{2\mu c}(S_x B_z - S_z B_x) \qquad \text{"Eq 2"}$$

and

$$\frac{dS_z}{dt} = \frac{eg}{2\mu c}(S_y B_x - S_x B_y) \qquad \text{"Eq 3"}$$

Combine Equations 1, 2, and 3 to see that

$$\boxed{\frac{d\vec{S}}{dt} = \frac{eg}{2\mu c}\vec{B}\times\vec{S}}$$

Let $\vec{B} = (0, 0, B)$.

Then $\dfrac{dS_x}{dt} = -\dfrac{eg}{2\mu c} B S_y$ "Eq 4"

$$\frac{dS_y}{dt} = \frac{eg}{2\mu c} B S_x \qquad \text{"Eq 5"}$$

$$\frac{dS_z}{dt} = 0$$

Combine Eq. 4 and Eq. 5 to get

$$\frac{d^2S_x}{dt^2} = -\left(\frac{egB}{2\mu c}\right)^2 S_x$$

Define $\dfrac{egB}{2\mu c} = \omega$

Then $S_x(t) = S_x(0)\cos\omega t + \dfrac{1}{\omega}\left(\dfrac{dS_x}{dt}\right)_0 \sin\omega t$

$$\boxed{S_x(t) = S_x(0)\cos\omega t + S_y(0)\sin\omega t}$$

Similarly, $\boxed{S_y(t) = S_y(0)\cos\omega t - S_x(0)\sin\omega t}$

and $\boxed{S_z(t) = S_z(0)}$

2 $\chi = \begin{pmatrix} i/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$

$$P = \left|(\chi_+^z)^\dagger \chi\right|^2 = \left|(1,0)\begin{pmatrix} i/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}\right|^2 = \left|\frac{i}{\sqrt{5}}\right|^2 = \boxed{\frac{1}{5}}$$

(3) A spin-3/2 particle can have $m_s = \{+\frac{3}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\}$

$$S_x = \frac{S_+ + S_-}{2}$$

$$S_+ |s, m_s\rangle = \hbar \sqrt{s(s+1) - m_s(m_s+1)}\, |s, m_s+1\rangle$$

$$S_- |s, m_s\rangle = \hbar \sqrt{s(s+1) - m_s(m_s-1)}\, |s, m_s-1\rangle$$

$S_x =$

final states ↓ / initial states ⇒	$\lvert\frac{3}{2}, +\frac{3}{2}\rangle$	$\lvert\frac{3}{2}, +\frac{1}{2}\rangle$	$\lvert\frac{3}{2}, -\frac{1}{2}\rangle$	$\lvert\frac{3}{2}, -\frac{3}{2}\rangle$
$\lvert\frac{3}{2}, +\frac{3}{2}\rangle$	0	$\frac{\hbar}{2}\sqrt{\frac{3}{2}(\frac{3}{2}+1) - \frac{1}{2}(\frac{1}{2}+1)}$	0	0
$\lvert\frac{3}{2}, +\frac{1}{2}\rangle$	$\frac{\hbar}{2}\sqrt{\frac{3}{2}(\frac{3}{2}+1) - \frac{3}{2}(\frac{3}{2}-1)}$	0	$\frac{\hbar}{2}\sqrt{\frac{3}{2}(\frac{3}{2}+1) - (-\frac{1}{2})(-\frac{1}{2}+1)}$	0
$\lvert\frac{3}{2}, -\frac{1}{2}\rangle$	0	$\frac{\hbar}{2}\sqrt{\frac{3}{2}(\frac{3}{2}+1) - \frac{1}{2}(\frac{1}{2}-1)}$	0	$\frac{\hbar}{2}\sqrt{\frac{3}{2}(\frac{3}{2}+1) - (-\frac{3}{2})(-\frac{3}{2}+1)}$
$\lvert\frac{3}{2}, -\frac{3}{2}\rangle$	0	0	$\frac{\hbar}{2}\sqrt{\frac{3}{2}(\frac{3}{2}+1) - (-\frac{1}{2})(-\frac{1}{2}-1)}$	0

So
$$\boxed{S_x = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2}\hbar & 0 & 0 \\ \frac{\sqrt{3}}{2}\hbar & 0 & \hbar & 0 \\ 0 & \hbar & 0 & \frac{\sqrt{3}}{2}\hbar \\ 0 & 0 & \frac{\sqrt{3}}{2}\hbar & 0 \end{pmatrix}}$$

$$S_y = \frac{S_+ - S_-}{2i}$$

$$S_y = \begin{array}{c|cccc} \text{initial states} \rightarrow & \left|\tfrac{3}{2},+\tfrac{3}{2}\right\rangle & \left|\tfrac{3}{2},+\tfrac{1}{2}\right\rangle & \left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle & \left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle \\ \text{final states}\downarrow & & & & \\ \hline \left|\tfrac{3}{2},+\tfrac{3}{2}\right\rangle & 0 & \frac{\sqrt{3}}{2}\frac{\hbar}{i} & 0 & 0 \\ \left|\tfrac{3}{2},+\tfrac{1}{2}\right\rangle & -\frac{\sqrt{3}}{2}\frac{\hbar}{i} & 0 & -i\hbar & 0 \\ \left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle & 0 & +i\hbar & 0 & \frac{\sqrt{2}}{2}\frac{\hbar}{i} \\ \left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle & 0 & 0 & -\frac{\sqrt{3}}{2}\frac{\hbar}{i} & 0 \end{array}$$

$$S_z |s, m_s\rangle = m_s \hbar \, |s, m_s\rangle$$

$$\boxed{S_z = \begin{pmatrix} \frac{3}{2}\hbar & 0 & 0 & 0 \\ 0 & \frac{1}{2}\hbar & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\hbar & 0 \\ 0 & 0 & 0 & -\frac{3}{2}\hbar \end{pmatrix}}$$

4 (a) $f_1 = A\sin\theta\, e^{i\phi}$

$$\langle f_1|f_1\rangle \equiv 1 = |A|^2 \int (\sin\theta\, e^{-i\phi})(\sin\theta\, e^{i\phi}) \sin\theta\, d\theta\, d\phi$$

$$= |A|^2 \int d\phi \int \sin^3\theta\, d\theta$$

$$= |A|^2 \cdot 2\pi \cdot \left[-\frac{1}{3}\cos\theta(\sin^2\theta + 2)\right]_0^{\pi}$$

$$= |A|^2 2\pi \left[\left(\frac{-1}{3}\right)(-1)(2) - \left(\frac{1}{3}\right)(1)(2)\right]$$

$$= |A|^2 \frac{8\pi}{3}$$

So $\boxed{A = \sqrt{\frac{3}{8\pi}}}$

$f_2 = B\cos\theta$

$$\langle f_2|f_2\rangle \equiv 1 = |B|^2 \int (\cos\theta)(\cos\theta)\sin\theta\, d\theta\, d\phi$$

$$= |B|^2 \cdot 2\pi \left[-\frac{\cos^3\theta}{3}\right]_0^{\pi}$$

$$= |B|^2 \cdot 2\pi \left(\frac{2}{3}\right)$$

So $\boxed{B = \sqrt{\frac{3}{4\pi}}}$

$f_3 = C\sin\theta\, e^{-i\phi}$

$\langle f_3|f_3\rangle = \langle f_1|f_1\rangle$, so $\boxed{C = A = \sqrt{\frac{3}{8\pi}}}$

(b) $-i\frac{d}{d\varphi}|f_1\rangle = -i\sqrt{\frac{3}{8\pi}}\frac{d}{d\varphi}\left(\sin\theta\, e^{i\varphi}\right) = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi} = |f_1\rangle$

$-i\frac{d}{d\varphi}|f_2\rangle = -i\sqrt{\frac{3}{4\pi}}\frac{d}{d\varphi}(\cos\theta) = 0$

$-i\frac{d}{d\varphi}|f_3\rangle = -i\sqrt{\frac{3}{4\pi}}\frac{d}{d\varphi}\left(\sin\theta\, e^{-i\varphi}\right) = -|f_1\rangle$

$\langle f_1|-i\frac{d}{d\varphi}|f_1\rangle = \langle f_1|f_1\rangle = 1$

$\langle f_1|-i\frac{d}{d\varphi}|f_2\rangle = 0$

$\langle f_1|-i\frac{d}{d\varphi}|f_3\rangle = -\langle f_1|f_1\rangle = -1$

$\langle f_2|-i\frac{d}{d\varphi}|f_1\rangle = \langle f_2|f_1\rangle = \int \cos\theta\sin\theta\, e^{i\varphi}\sin\theta\, d\theta\, d\varphi = 0$

$\langle f_2|-i\frac{d}{d\varphi}|f_2\rangle = 0$

$\langle f_2|-i\frac{d}{d\varphi}|f_3\rangle = -\langle f_2|f_3\rangle = 0$

$\langle f_3|-i\frac{d}{d\varphi}|f_1\rangle = \langle f_3|f_1\rangle = \int \sin\theta\, e^{i\varphi}\sin\theta\, e^{i\varphi}\sin\theta\, d\theta\, d\varphi = 0$

$\langle f_3|-i\frac{d}{d\varphi}|f_2\rangle = 0$

$\langle f_3|-i\frac{d}{d\varphi}|f_3\rangle = -\langle f_3|f_1\rangle = 0$