Quantum Mechanics 2 Homework #5

1) Consider a positronium atom that consists of an electron and a positron (a positively charged electron) in a hydrogen-like bound state. No nucleus is present. Write down the Hamiltonian for this system in the presence of a constant external magnetic field. Show that (ignoring spin) this system experiences no Zeeman effect.

2) Consider a particle of mass M attached to a rigid massless rod of fixed length R whose other end is fixed at the origin. The rod is free to rotate about its fixed point.

(a) Write down the Hamiltonian for this system (hint: recall the rigid rotator of Chapter 11.)

(b) If the particle carries a charge Q and rotor is placed in a constant magnetic field $\vec{B}$, what is the modified Hamiltonian?

3) Consider a charged particle moving in a uniform magnetic field. If the field has a magnitude of 10^4 gauss, what type of radiation (x-rays? microwaves?) does the particle emit?

4) Goswami problem 14.6.

5) Consider a charged particle in a magnetic field $\vec{B} = (0,0,B)$ and an electric field $\vec{E} = (E,0,0)$. Choose $\vec{A} = (0,Bx,0)$.

(a) Confirm that this choice of $\vec{A}$ is consistent with the given form of $\vec{E}$.

(b) Find the eigenvalues and eigenvectors for this system.

(1) H for 1 charged particle $q=-e$ in a general Em field is

$$H = \frac{\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2}{2m} + e\phi$$

For 2 particles with $q = +e$ and $-e$, this becomes

$$H = \frac{\left(\vec{p}_1 + \frac{e\vec{A}}{c}\right)^2}{2m_1} + \frac{\left(\vec{p}_2 - \frac{e\vec{A}}{c}\right)^2}{2m_2} + e\phi$$

Recall when the magnetic field is uniform we can write

$$\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$$

Since a Coulomb field is present,

$$\phi = \frac{-e}{|\vec{r}_1 - \vec{r}_2|}$$

So we have

$$\boxed{H = \frac{\left[\vec{p}_1 + \frac{e}{2c}\vec{B}\times\vec{r}_1\right]^2}{2m_1} + \frac{\left[p_2 - \frac{e}{2c}\vec{B}\times\vec{r}_2\right]^2}{2m_2} - \frac{e^2}{|\vec{r}_1 - \vec{r}_2|}}$$

To see that no normal Zeeman effect occurs, define:

$\vec{P} \equiv \vec{p}_1 + \vec{p}_2$ (center-of-mass momentum)

$\vec{p} \equiv \frac{\vec{p}_1 - \vec{p}_2}{2}$ (relative momentum)

$\vec{R} \equiv \frac{\vec{r}_1 + \vec{r}_2}{2}$ (center-of-mass position)

$\vec{r} = \vec{r}_1 - \vec{r}_2$ (relative position)

Then $\vec{p}_1 = \frac{\vec{P}}{2} + \vec{p}$

$\vec{p}_2 = \frac{\vec{P}}{2} - \vec{p}$

$\vec{r}_1 = \vec{R} + \frac{\vec{r}}{2}$

$\vec{r}_2 = \vec{R} - \frac{\vec{r}}{2}$

Recall that the Zeeman Effect is due to a term given for one particle as $H_{int} = \frac{e}{2mc} B L_z$ (Goswami Eq. 14.14)

Here for 2 particles that term becomes

$B[\vec{p}_1 \times \vec{r}_1 - \vec{p}_2 \times \vec{r}_2]$

$= B[2\vec{p} \times \vec{R} + \frac{1}{2}\vec{P} \times \vec{r}]$

$\boxed{= 0}$ because $\vec{R}$ can be chosen $= 0$ and $\vec{P} = 0$ in the absence

(2) (a) $H = \frac{L^2}{2I} = \frac{(\vec{R}\times\vec{p})^2}{MR^2}$

(b) Recall that if the charge experiences a vector potential $\vec{A}$, the Hamiltonian must be modified according to

$$\vec{p} \rightarrow \vec{p} - \frac{q}{c}\vec{A}.$$

Then if $\vec{B}$ is constant we can write

$$\vec{A} = \frac{1}{2}\vec{B}\times\vec{R}$$

So $\vec{R}\times\vec{p}$ becomes $\vec{R}\times\left[\vec{p} - \frac{q}{2c}(\vec{B}\times\vec{R})\right]$

$$= \vec{R}\times\vec{p} - \frac{q}{2c}\vec{R}\times(\vec{B}\times\vec{R})$$

$$= \vec{L} - \frac{q}{2c}\left[BR^2 - R(\vec{B}\cdot\vec{R})\right]$$

So $$\boxed{H = \frac{\left[\vec{L} - \frac{q}{2c}\left[BR^2 - R(\vec{B}\cdot\vec{R})\right]\right]^2}{MR^2}}$$

(3) Consider only the effect of $H_1 = \frac{e}{2\mu c} B L_z$

This splits lines according to their m values,

$$E_1 = \frac{eB\hbar m}{2\mu c}$$

Suppose that a transition between the $m=1$ and $m=0$ levels is possible. Then the emitted photon will have

$$E = \frac{eB\hbar}{2\mu c} = 5.7\times10^{-9} B$$

When $B = 10^4$ gauss

$$E = 5.7\times10^{-5}\,\text{eV} = h\nu$$

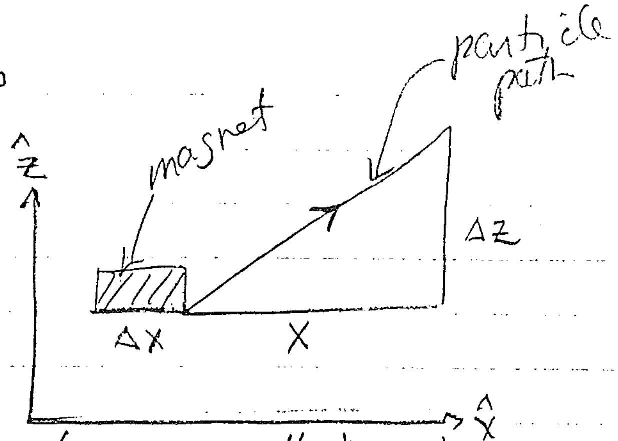
$$\nu = 1.4\times10^{10}\,\text{Hz}$$

(4) $\mu_z = \frac{e\hbar}{2m_e c} = 5.8 \times 10^{-9}$ eV/gauss

$T = 0.2$ eV

$\Delta x = 3$ cm

$\frac{\Delta B}{\Delta z} = 200 \times 10^3$ gauss/cm



(ignore deflection of path inside the magnet)

(a) Find $\Delta z (x = 20\text{ cm})$

$$F_z = \mu_z \frac{dB}{dz}$$

$$= 5.8 \times 10^{-9} \frac{eV}{gauss} \times 2 \times 10^5 \frac{gauss}{cm} = 1.16 \times 10^{-3} \frac{eV}{cm}$$

The time during which the force is exerted is

$$\Delta t = \frac{\Delta x}{v_x} = \frac{\Delta x}{\sqrt{2T/m}}$$

After going through the magnet,

$$v_z = \frac{F_z \Delta t}{m} = \frac{F_z \Delta x}{m\sqrt{T/2m}}$$

So $\Delta z = \frac{x}{v_x} \cdot v_z = \frac{x \cdot \frac{F_z \Delta x}{m\sqrt{2T/m}}}{\sqrt{2T/m}} = \frac{x F_z \Delta x}{2T} = \boxed{0.17 \text{ cm}}$

(b) Deflection by the Lorentz Force would be

$$\Delta z_{Lorentz} = \frac{x F_{Lorentz}}{2T} \Delta x = \frac{x q v_x B \Delta x}{2T}$$

$$= \frac{xq \sqrt{2T/m}\, B \Delta x}{2T} = \boxed{\frac{qBx\Delta x}{\sqrt{2Tm}}}$$

⑤ (a) $\nabla \times A = \hat{x}\left(\frac{dA_z}{dy} - \frac{dA_y}{dz}\right) + \hat{y}\left(\frac{dA_x}{dz} - \frac{dA_z}{dx}\right) + \hat{z}\left(\frac{dA_y}{dx} - \frac{dA_x}{dy}\right)$

$= \hat{x}\left(0 - \frac{d}{dz}(Bx)\right) + \hat{y}\left(0 - 0\right) + \hat{z}\left(\frac{d}{dx}(Bx) - 0\right) = \boxed{B\hat{z}}$

(b) $H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 - qE \cdot \vec{r}$

Plug in $\vec{E} = E\hat{x}$ and $\vec{B} = B\hat{z}$ and $A = Bx\hat{y}$

$$H = \frac{1}{2m}\left[p_x^2 + \left(p_y + \frac{eBx}{c}\right)^2 + p_z^2\right] + eEx$$

$$= \frac{1}{2m}\left[p_x^2 + p_y^2 + \left(\frac{2eBp_y}{c} + eE\right)x + p_z^2 + \frac{e^2B^2x^2}{c^2}\right]$$

Require that the eigenstate be a simultaneous eigenstate of H, p_y (with eigenvalue $\hbar k$) and p_z (with eigenvalue 0).

Then $H = \frac{p_x^2}{2m} + \frac{\hbar^2 k^2}{2m} + \frac{1}{2m}\left(\frac{2eB\hbar k}{c} + eE\right)x + \frac{e^2B^2x^2}{2mc^2}$

This can be shown to be the harmonic oscillator hamiltonian (with equilibrium point offset from 0) by rewriting it as:

$$H = \frac{\hbar^2 k^2}{2m} + \frac{p_x^2}{2m} + \frac{e^2B^2}{2mc^2}\left[x^2 + 2x\left(\frac{\hbar kc}{eB} - \frac{eEc^2}{e^2B^2}\right)\right]$$

$$= \frac{\hbar^2 k^2}{2m} + \frac{p_x^2}{2m} + \frac{e^2B^2}{2mc^2}\left[x + \frac{\hbar kc}{eB} - \frac{Ec^2}{eB^2}\right]^2 + \text{Constant}$$

let $\omega \equiv \frac{e^2B^2}{mc^2}$

and $x' = x + \frac{\hbar kc}{eB} - \frac{Ec^2}{eB^2}$

Then $H = \frac{\hbar^2 k^2}{2m} + \frac{p_x^2}{2m} + \frac{m\omega^2 x'^2}{2}$

So the e'values + e'vectors are the same as for the harmonic oscillator

(iii) 3^{RD} state : $n=2$

$$E_2^{(1)} = 39C(\hbar/2m\omega)^2$$

$$E_2^{(2)} = \frac{-615C^2\hbar^3}{8m^4\omega^5}$$