Quantum Mechanics 2 Homework #4

1) Goswami problem 13.5.

2) Goswami problem 13.6.

3) An electron in the Coulomb field of a proton is in a state described by the wavefunction

$$\psi = \frac{1}{6}\left[4\psi_{100} + 3\psi_{211} - \psi_{210} + \sqrt{10}\psi_{21-1}\right].$$

(a) What is the expectation value of the energy?

(b) What is the expectation value of L^2?

(c) What is the expectation value of L_z?

4) Construct the spatial wavefunction $\psi(n = 4,\ \ell = 2,\ m = 1)$ for hydrogen. Express your answer as a function of r, θ, ϕ, and a_0 only. (That is, you may not use other variables such as ρ or z, functions such as the Y_ℓ^m or Laguerre polynomials, or constants other than π, e, and numerals.

5) Calculate the probability current $\vec{J}$ of a system whose Hamiltonian is given by Goswami Equation 14.7. Be sure that probability remains conserved.

QM 2

Answers to homework 4

(1) $V(r) = -\frac{e^2}{r}\left(1 + \frac{\alpha}{r}\right)$

Plugging this into the Radial Equation (Form 1) gives:

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)R + \frac{2\mu}{\hbar^2}\left(E + \frac{e^2}{r^2} - \frac{(-2\mu\alpha e^2 + \hbar^2 l\{l+1\})}{2\mu r^2}\right)R = 0$$

Define $l'(l'+1) \equiv \frac{-2\mu\alpha e^2}{\hbar^2} + l(l+1)$

Then the equation is identical to Goswami Eq. 13.1, with $l \to l'$ and $Z \to 1$

Thus the solution of Goswami Eq. 1 applies

One gets (Goswami Eq. 13.11) $\lambda = n_r + l' + 1 = n$

Notice $n_r + 1 = n - l$, so

$$\lambda = n - l + l'$$

Then $E_n = -\frac{\mu}{2}\left(\frac{Ze^2}{\hbar\lambda}\right)^2 = \frac{-\mu Z^2 e^4}{2\hbar^2(n - l + l')^2}$

$$= \frac{-\mu Z^2 e^4}{2\hbar^2 n^2}\left[1 - \frac{1}{n}\left(l + \frac{1}{2} - \sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{2\mu\alpha e^2}{\hbar^2}}\right)\right]^{-2}$$

(2) For ψ_{100}, $\langle r \rangle = \int_0^\infty r^2 dr\, R_{10}^* \, r \, R_{10} = \frac{3a_0}{2}$

For ψ_{210}, $\langle r \rangle = \int_0^\infty r^2 dr\, R_{21}^* \, r \, R_{21} = 5a_0$

Prob. is a maximum at the r for which $\frac{d\,Prob}{dr} = 0$

For ψ_{100}, $Prob = r^2 |R_{10}|^2 = \frac{4}{a_0^3} r^2 e^{-2r/a_0} \rightarrow r_{max} = a_0$

For ψ_{210}, $Prob = r^2 |R_{21}|^2 = \frac{1}{3} \cdot \frac{1}{8a_0^3} \frac{1}{a_0^2} r^4 e^{-r/a_0} \longrightarrow r_{max} = 4a_0$

For the proof, recall Goswami Eq. 13.21 for $l = n-1$:

$$Prob(r) \propto r^2 r^{2n-2} e^{-2r/na_0} \underbrace{\left[L_{2n-1}^{2n-1} \right]}_{(2n-1)!\text{, a constant}}$$

$$\frac{d\,Prob}{dr} = r^{2n}\left(\frac{-2}{na_0}\right) e^{-2r/na_0} + (2n) r^{2n-1} e^{-2r/na_0} = 0 \text{ at max}$$

So $r_{max}\left(\frac{-2}{na_0}\right) + (2n) = 0$

$$r_{max} = \frac{(2n)na_0}{2} = \boxed{n^2 a_0}$$

③ ⓐ $\langle\psi|E|\psi\rangle = \left(\frac{4}{6}\right)^2 E_1 + \left(\frac{3}{6}\right)^2 E_2 + \left(\frac{-1}{6}\right)^2 E_2 + \left(\frac{\sqrt{10}}{6}\right)^2 E_2$ ③

$$E_i = \frac{-Z^2 e^4 \mu}{2\hbar^2 n^2} = \frac{13.3\,eV}{n^2} \quad \text{for } Z=1$$

So $E_1 = 13.3\,eV$
$E_2 = 3.325\,eV$

$\langle\psi|E|\psi\rangle = \boxed{7.76\,eV}$

ⓑ $\langle\psi|L^2|\psi\rangle = \hbar^2\left[\left(\frac{4}{6}\right)^2\cdot 0 + \left(\frac{3}{6}\right)^2\left(1(1+2)\right) + \left(\frac{-1}{6}\right)^2\left(1(1+2)\right)\right.$

$\left. + \left(\frac{\sqrt{10}}{6}\right)^2\left(1(1+2)\right)\right] = \boxed{\frac{10\hbar^2}{9}}$

ⓒ $\langle\psi|L_z|\psi\rangle = \hbar\left[\left(\frac{4}{6}\right)^2\cdot 0 + \left(\frac{3}{6}\right)^2\cdot 1 + \left(\frac{-1}{6}\right)^2\cdot 0 + \left(\frac{\sqrt{10}}{6}\right)^2\cdot(-1)\right] = \boxed{\frac{-\hbar}{36}}$

(4) $\psi_{421} = R_{42} Y_2^1$

From Goswami Eq 13.21,

$$R_{42} = -\left[\left(\frac{2Z}{4a_0}\right)^3 \frac{(4-2-1)!}{2\cdot 4[(4+2)!]^3}\right]^{1/2} \left(\frac{2\cdot Z r}{4a_0}\right)^2 e^{-Zr/4a_0} L_6^5$$

$$L_6^5(x) = \frac{d^5}{dx^5} L_6(x)$$

$$L_6(x) = e^x \frac{d^6}{dx^6}\left(x^6 e^{-x}\right)$$

$Z = 1$

So $$R_{42} = -\left[\frac{1}{(8a_0^3)} \frac{1}{5760}\right]^{1/2} \left(\frac{r^2}{4a_0^2}\right) e^{-r/4a_0} \left[-4320 + 720\left(\frac{r}{2a_0}\right)\right]$$

From Goswami Eq. 11.40,

$$Y_2^1 = \left[\frac{5}{4\pi}\frac{(1)!}{(3)!}\right]^{1/2}(-1)\,e^{im\varphi}\,P_2^1$$

$$P_2^1(\xi) = (1-\xi^2)^{1/2}\frac{d}{d\xi}P_2$$

$$= (1-\xi^2)^{1/2}\frac{d}{d\xi}\left[\frac{1}{2}(3\xi^2-1)\right]$$

$$= \frac{(1-\xi^2)^{1/2}}{2}(6\xi)$$

$$= 3\xi(1-\xi^2)^{1/2}$$

Since $\xi = \cos\theta$,

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\,e^{i\varphi}\,\frac{\sin 2\theta}{2}$$

(5) Recall for the case where $\vec{A}=0$, $\vec{J}$ was defined by

$$\vec{J} = \frac{\hbar}{2mi}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right)$$

$$= \frac{1}{2m}\left(\Psi^* p_{op}\Psi - \Psi p_{op}\Psi^*\right)$$

Goswami Eq. 14.6, and the handout "Appendix E" from Townsend show that inclusion of $\vec{A}$ requires changing

$p \rightarrow p + \frac{e\vec{A}}{c}$ in the Hamiltonian.

If we carry over that change to $\vec{J}$, $\vec{J}$ becomes

$$J = \frac{1}{2m}\left[\psi^*\left(p_{op} + \frac{e}{c}A_{op}\right)\psi - \psi\left(p_{op} + \frac{e}{c}A_{op}\right)\psi^*\right]$$

$$\boxed{= \frac{1}{2m}\left[\psi^* p_{op}\psi - \psi p_{op}\psi^* + \frac{e}{c}A_{op}\psi^*\psi\right]}$$

To guarantee conservation of probability, we still require that

$$\nabla\cdot J + \frac{\partial\rho}{\partial t} = 0$$

Recall $\rho = \psi^*\psi$ (Gasiorowicz Eq. 1.27)

So the inclusion of $\vec{A}$ transforms J to $J_{old} + \frac{e}{c}A\psi^*\psi$

(Some students preferred to work out this problem for the special case of $\psi = \psi_{nmk}$ in the static, uniform $\vec{B}$ field special case; that's ok)