Quantum Mechanics 2 Homework #3

1) Goswami problem 12.8.

2) Assume that the deuteron is a bound state with $\ell = 0$, and the potential is a square well of range $r = 2.8 \times 10^{-13}$ cm. Given that the binding energy is $-2.18$ MeV, find the depth of the potential. Here is a hint about how to do this: first convert distances and masses into units of the reduced mass $\mu$, so that the range is given in units of $\hbar / \mu c$ and the binding energy in units of $\mu c^2$. Notice that the binding energy is quite close to zero, so you can expand the potential around that for which the binding energy is zero.

3) Write down the eigenvalue condition for a square well potential of range $a$ and depth $V_0$, for $\ell = 1$. The only symbols allowed in the equation are $\mu$, $V_0$, $E$, $a$, fundamental constants, and trigonometric functions.

4) Write down the three-dimensional Schroedinger Equation for the harmonic oscillator potential. Separate the equation in spherical coordinates. Use the power series method to solve the radial equation. Find the recursion relation for the coefficients and determine the allowed energies.

5) Find the matrix representations of the operators $L_x$, $L_y$, and $L_z$ in the basis $|\ell, m_\ell\rangle$, when $\ell = 1$. Arrange the rows and columns of the matrices in order of decreasing $m_\ell$ value. By explicit multiplication of the matrices, show that $\left[L_x, L_y\right] = i\hbar L_z$.

QM 2

ANSWERS TO HOMEWORK 3

(1) $P(r) = r^2 R^2$

First normalize the deuteron wave function:

$$1 \equiv \int_0^\infty r^2 |R|^2 dr$$

$$= |A|^2 \int_0^a \sin^2 k_{in} r \, dr + |C|^2 \int_a^\infty \exp(-2k_B r) \, dr$$

$$= |A|^2 \left[ \left( \frac{a}{2} - \frac{1}{2k_{in}} \sin(k_{in}a)\cos(k_{in}a) \right) + \frac{|C|^2}{|A|^2} \frac{1}{2k_B} \exp(-2k_B a) \right]$$

Requiring continuity of $\psi$ @ $r=a$ forces

$$\left|\frac{C}{A}\right|^2 = \sin^2(k_{in}a) \exp(2k_B a)$$

Also plug in (Goswami Eq. 12.37):

$$k_{in} \cot(k_{in}a) = -k_B$$

Then $A = \pm \sqrt{\frac{2}{a}} \left(1 + \frac{1}{k_B a}\right)^{-1/2}$ (choose + sign)

so $C = \sqrt{\frac{2}{a}} \left(1 + \frac{1}{k_B a}\right)^{-1/2} \sin(k_{in}a) \exp(k_B a)$

So $u = rR = \begin{cases} \sqrt{\frac{2}{a}} \left(1 + \frac{1}{k_B a}\right)^{-1/2} \sin k_{in} r & \text{for } r < a \\ \sqrt{\frac{2}{a}} \left(1 + \frac{1}{k_B a}\right)^{-1/2} \sin(k_{in}a) \exp(k_B a) \exp(-k_B r) & \text{for } r > a \end{cases}$

Now we need $k_B a$ and $k_{in} a$:

$$k_B a = a\sqrt{\frac{2\mu|E|}{\hbar^2}} = \frac{a}{c\hbar}\sqrt{m_n c^2 |E|}$$

$$= \frac{1.4\times10^{-15}\,\text{m}\sqrt{(940\,\text{MeV})(2.23\,\text{MeV})}}{(3\times10^{8}\,\text{m/s})(6.58\times10^{-22}\,\text{MeV-s})} = \underline{0.325}$$

To find $k_{in}a$, solve Goswami Eq 12.41:

$$k_{in}a = \arctan\left(\frac{-k_{in}a}{k_B a}\right) = \arctan\left(\frac{-k_{in}a}{0.325}\right)$$

$k_{in}a = 1.754$ for the ground state

Plug these into $u(r) = rR$, then

$$\text{Prob} = \begin{cases} \dfrac{2}{1.4\times10^{-15}\,\text{m}}\left(1+\dfrac{1}{0.325}\right)^{-1}\sin^2\left(\dfrac{1.754\,r}{1.4\times10^{-15}\,\text{m}}\right) & r<a \\ 0.35\cdot\sin^2(1.754)\exp(0.650)\exp\left(\dfrac{-0.650\,r}{1.4\times10^{-15}\,\text{m}}\right) & r>a \end{cases}$$

$$\boxed{= \begin{cases} 0.35\sin^2(1.25r) & r<a \\ 0.65\exp(-0.464r) & r>a \end{cases}}$$

3) Suppose that $V_1$ is the potential depth that corresponds to zero energy. It is specified by

"Call this Eq 1" → $$\frac{2mV_1a^2}{\hbar^2} = \frac{\pi^2}{4}$$ (eq. following Goswami Eq. 12.39)

Suppose that the real depth of the deuteron potential is $V_0$, which differs slightly (by $\Delta V$) from $V_1$, so

$$V_0 = V_1 + \Delta V$$

Convert the V's to dimensionless U's:

$$V_1 \equiv mc^2 U_1$$
$$\Delta V \equiv mc^2 \Delta U$$

Convert the well width (range) "a" to unitless form "d":

$$a \equiv \frac{\hbar}{mc} d$$

Also scale the energy to unitless form "$\mathcal{E}$"

$$|E| \equiv \mathcal{E} mc^2$$

Plug all of this into Eq 1:

$$\frac{2m\,mc^2 U_1}{\hbar^2}\frac{\hbar^2 d^2}{m^2c^2} = \frac{\pi^2}{4}$$

$$\downarrow$$

$$U_1 d^2 = \frac{\pi^2}{8}$$

As was done in class, we can define

$$\lambda \equiv \frac{2mV_0a^2}{\hbar^2} \quad \text{and}$$

$y = ka$, where

$$k \equiv \sqrt{\frac{2m(V_0 - |E|)}{\hbar^2}}$$

We need to plug into

call this Eq 2'' $\rightarrow$ $\frac{\sqrt{\lambda - y^2}}{y} = \tan y$, the transcendental equation for the energy

Plug in:

$$\lambda = \frac{2m}{\hbar^2} mc^2(U_1 + \Delta U)\frac{\hbar^2 d^2}{m^2c^2} = 2d^2(U_1 + \Delta U)$$

$$= \frac{\pi^2}{4} + 2d^2\Delta U$$

and

$$y^2 = \frac{2m(V_0 - |E|)a^2}{\hbar^2} \rightarrow \frac{\pi^2}{4} + 2d^2(\Delta U - \mathcal{E})$$

$$y = \sqrt{y^2} = \left[\frac{\pi^2}{4} + 2d(\Delta U - \mathcal{E})\right]^{1/2}$$

Expand this in a series, since $\Delta U$ and $\mathcal{E}$ are small

$$y \approx \frac{\pi}{2} + \frac{4d^2(\Delta U - \varepsilon)}{\pi}$$

Then Eq 2 becomes

$$\frac{\sqrt{2\varepsilon d^2}}{\frac{\pi}{2} + \frac{2d^2(\Delta U - \varepsilon)}{\pi}} = \tan\left(\frac{2d^2(\Delta U - \varepsilon)}{\pi}\right)$$

If $\frac{2d^2(\Delta U - \varepsilon)}{\pi} \ll \frac{\pi}{2}$, then the LHS denominator simplifies, and we have

$$\frac{2d^2(\Delta U - \varepsilon)}{\pi} = \arctan\sqrt{\frac{8\varepsilon d^2}{\pi^2}}$$

$$\Delta U = 6.25 \times 10^{-3}$$

So $V_0 = 59$ meV

(3) Begin with Goswami Eq. 12.50:

$$k_I\left[\frac{dj_\ell(\rho)/d\rho}{j_\ell(\rho)}\right]_{\rho=k_I a} = ik_{II}\left[\frac{dh_\ell^{(1)}/d\rho}{h_\ell^{(1)}(\rho)}\right]_{\rho=k_{II}a}$$

$$k_I = \frac{\sqrt{2\mu(V_0-|E|)}}{\hbar}$$

$$k_{II} = \frac{\sqrt{2\mu|E|}}{\hbar}$$

$\ell = 1$

$$j_1(\rho) = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho}$$

$$\frac{dj_1(\rho)}{d\rho} = \frac{-2\sin\rho}{\rho^3} + \frac{2\cos\rho}{\rho^2} + \frac{\sin\rho}{\rho}$$

Plug in $\rho = \frac{a\sqrt{2\mu(V_0-|E|)}}{\hbar}$

$$h_1^{(1)}(\rho) = j_1(\rho) + in_1(\rho)$$

$$= \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho} - \frac{i\cos\rho}{\rho^2} - \frac{i\sin\rho}{\rho} = \frac{1}{\rho^2}e^{-i\rho} - \frac{1}{\rho}e^{+i\rho}$$

$$\frac{dh_1^{(1)}(\rho)}{d\rho} = -\frac{2e^{-i\rho}}{\rho^3} - \frac{i}{\rho^2}e^{-i\rho} + \frac{e^{i\rho}}{\rho^2} - \frac{ie^{i\rho}}{\rho}$$

Plug in $\rho = \frac{a\sqrt{2\mu|E|}}{\hbar}$

Plug all this into the equation at the top of the page.

4 $$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[\frac{m\omega^2 r^2}{2} + \frac{\hbar^2 l(l+1)}{2mr^2}\right]u = Eu$$

Let $\xi = \sqrt{\frac{m\omega}{\hbar}}\, r$ and $K \equiv \frac{2E}{\hbar\omega}$

Then we have

$$-\frac{\hbar^2}{2m}\frac{m\omega}{\hbar}\frac{d^2u}{d\xi^2} + \left[\frac{m\omega^2}{2}\frac{\hbar}{m\omega}\xi^2 + \frac{\hbar^2}{2m}\frac{m\omega}{\hbar}\frac{l(l+1)}{\xi^2}\right]u = Eu$$

$$\downarrow$$

$$\frac{d^2u}{d\xi^2} = \left[\xi^2 + \frac{l(l+1)}{\xi^2} - K\right]u \qquad \text{(Call this "Eq 1")}$$

For large $\xi$, $\frac{d^2u}{d\xi^2} \approx \xi^2 u$; then $u \sim e^{-\xi^2/2}$

For small $\xi$, $\frac{d^2u}{d\xi^2} \approx \frac{l(l+1)u}{\xi^2}$; then $u \sim \xi^{l+1}$

So guess the general solution to $u$ is

$u(\xi) = \xi^{l+1} e^{-\xi^2/2} v(\xi)$, and solve for $v(\xi)$ by plugging into Eq 1.

$$\frac{du}{d\xi} = (l+1)\xi^{l} e^{-\xi^2/2} v - \xi^{l+2} e^{-\xi^2/2} v + \xi^{l+1} e^{-\xi^2/2} v'$$

$$\begin{aligned}\frac{d^2u}{d\xi^2} = {} & l(l+1)\xi^{l-1} e^{-\xi^2/2} v - (l+1)\xi^{l+1} e^{-\xi^2/2} v + (l+1)\xi^{l} e^{-\xi^2/2} v' \\ & -(l+2)\xi^{l+1} e^{-\xi^2/2} v + \xi^{l+3} e^{-\xi^2/2} v - \xi^{l+2} e^{-\xi^2/2} v' \\ & +(l+1)\xi^{l} e^{-\xi^2/2} v' - \xi^{l+2} e^{-\xi^2/2} v' + \xi^{l+1} e^{-\xi^2/2} v''\end{aligned}$$

Plug $u$, $\frac{du}{d\xi}$, and $\frac{d^2u}{d\xi^2}$ into Eq. 1 and cancel terms. You get:

$$v'' + 2v'\left(\frac{\ell+1}{\xi} - \xi\right) + (K - 2\ell - 3)\, v = 0 \qquad (\text{"Eq 2"})$$

Let $v(\xi) = \sum_{j=0}^{\infty} a_j \xi^j$.

Then $v' = \sum_{j=1}^{\infty} j a_j \xi^{j-1}$ and

$$v'' = \sum_{j=2}^{\infty} j(j-1) a_j \xi^{j-2}$$

Plug these into Eq 2:

$$\sum_{j=2}^{\infty} j(j-1) a_j \xi^{j-2} + 2(\ell+1)\sum_{j=1}^{\infty} j a_j \xi^{j-2} - 2\sum_{j=1}^{\infty} j a_j \xi^j + (K-2\ell-3)\sum_{j=0}^{\infty} a_j \xi^j =$$

In the first 2 sums, rename $j \to j+2$ (dummy variable).

We can also begin the third sum at $j=0$ instead of $j=1$

with no effect.

$$\sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2}\xi^j + 2(\ell+1)\sum_{j=-1}^{\infty} (j+2) a_{j+2} \xi^j - 2\sum_{j=0}^{\infty} j a_j \xi^j$$

$$+ (K-2\ell-3)\sum_{j=0}^{\infty} a_j \xi^j = 0.$$

The only way to satisfy this equation, including the $j=-1$ term in the second sum, is for $\underline{a_1 = 0}$. Apply this + combine term

$$\sum_{j=0}^{\infty} \left\{ (j+2)(j+2l+3)a_{j+2} + (K-2j-2l-3)a_j \right\} \xi^j = 0, \text{ so}$$

$$a_{j+2} = \frac{(2j+2l-3-K)}{(j+2)(j+2l+3)} a_j$$

To force the serie to terminate, choose $K = 2j_{max} + 2l + 3$.

This make $a_{j_{max}+2} = 0$.

But $E = \frac{\hbar\omega K}{2}$, so

$$E = \left(j_{max} + l + \frac{3}{2}\right)\hbar\omega.$$

Rename $j_{max} + l \equiv n$.

Then $\boxed{E = \left(n + \frac{3}{2}\right)\hbar\omega}$ $n = 1, 2, 3, \ldots$

5) $L_x = \dfrac{L_+ + L_-}{2}$

$$\langle l'm' | L_x | l,m\rangle = \frac{1}{2}\langle l'm' | L_+ | lm\rangle + \frac{1}{2}\langle l'm' | L_- | lm\rangle$$

$$= \frac{\hbar}{2}\sqrt{l(l+1)-m(m+1)}\,\delta_{l'l}\,\delta_{m',m+1} + \frac{\hbar}{2}\sqrt{l(l+1)-m(m-1)}\,\delta_{l'l}\,\delta_{m',m-1}$$

| $\langle l'm'\| \Downarrow$ \ $\|lm\rangle \Rightarrow$ | $\|1,1\rangle$ | $\|1,0\rangle$ | $\|1,-1\rangle$ |
|---|---|---|---|
| $\langle 1,1\|$ | 0 | $\hbar\sqrt{2}/2$ | 0 |
| $\langle 1,0\|$ | $\hbar\sqrt{2}/2$ | 0 | $\hbar 2\sqrt{2}$ |
| $\langle 1,-1\|$ | 0 | $\hbar\sqrt{2}/2$ | 0 |

$L_y = \dfrac{L_+ - L_-}{2i}$

$$\langle l'm' | L_y | lm\rangle = \frac{1}{2i}\langle l'm' | L_+ | lm\rangle - \frac{1}{2i}\langle l'm' | L_- | lm\rangle$$

$$= \frac{\hbar}{2i}\sqrt{l(l+1)-m(m+1)}\,\delta_{l'l}\,\delta_{m',m+1} - \frac{\hbar}{2i}\sqrt{l(l+1)-m(m-1)}\,\delta_{l'l}\,\delta_{m',m-1}$$

| | $\|1,1\rangle$ | $\|1,0\rangle$ | $\|1,-1\rangle$ |
|---|---|---|---|
| $\langle 1,1\|$ | 0 | $\hbar\sqrt{2}/2i$ | 0 |
| $\langle 1,0\|$ | $-\hbar\sqrt{2}/2i$ | 0 | $\hbar\sqrt{2}/2i$ |
| $\langle 1,-1\|$ | 0 | $-\hbar\sqrt{2}/2i$ | 0 |

$\langle l'm' | L_z | lm \rangle = m\hbar\, \delta_{l'l}\, \delta_{m'm}$

$$\begin{array}{c|ccc} & |1,1\rangle & |1,0\rangle & |1,-1\rangle \\ \hline \langle 1,+1| & \hbar & 0 & 0 \\ \langle 1,0| & 0 & 0 & 0 \\ \langle 1,-1| & 0 & 0 & -\hbar \end{array}$$

The relation $[L_x, L_y] = i\hbar L_z$ follows directly