Quantum Mechanics 2 Homework #2

1) Goswami problem 12.10.

2) At a time $t = 0$, a particle is in the superposition state: $\psi(\vec{r},0) = \frac{1}{2\pi^{3/2}} \sin(3x)\exp[i(5y + z)]$

(a) If the energy is measured at $t = 0$, what value will be found?

(b) What possible values of momentum (p_x, p_y, p_z) will measurement find?

(c) Given the above expression for $\psi(\vec{r},0)$, what is $\psi(\vec{r},t)$?

(d) If p is measured at $t = 0$ and the value $\vec{p} = \hbar(3\hat{x} + 5\hat{y} + \hat{z})$ is found, what is $\psi(\vec{r},t)$?

3) The relativistic analog of the Schroedinger Equation for a spin-zero (i.e., non-physical) electron is

$\left(\frac{E}{\hbar c} + \frac{Ze^2}{\hbar c r}\right)\psi = -\nabla^2\psi + \left(\frac{mc}{\hbar}\right)^2 \psi$. Find the radial equation.

4) Consider a particle trapped in a cylindrical well of radius a. The well is unbounded in z. The particle is consequently subject to the condition: $V(\rho) = \begin{cases} 0 & \text{if } \rho < a \\ \infty & \text{if } \rho \geq a \end{cases}$

(a) Write down the Schroedinger Equation in cylindrical coordinates for this particle.

(b) Guess that the equation can be solved by a wavefunction of the form $\psi = R(\rho)\Phi(\phi)$. No conditions are applied in z.)

(c) Find equations for $R(\rho)$ and $\Phi(\phi)$.

(d) List the boundary conditions to which $R(\rho)$ and $\Phi(\phi)$ are subject.

(e) Solve the equations if they are familiar to you.

5) Goswami problem 12.A1.

QM 2

ANSWERS TO HOMEWORK 2

1) $\frac{d^2u}{dr^2} + \frac{2\mu}{\hbar^2}\left[E - V - \frac{\hbar^2 l(l+1)}{2\mu r^2}\right]u = 0$ Plug in $V = -V_0 e^{-2\beta r}$ and $l = 0$

$$\frac{d^2u}{dr^2} + \frac{2\mu}{\hbar^2}\left[E + V_0 \exp(-2\beta r)\right]u = 0$$

Let $\rho = \exp(-\beta r)$

Then you get $\frac{d^2u}{d\rho^2} + \frac{du}{\rho d\rho} + \left(\frac{2\mu V_0}{\beta^2\hbar^2} - \frac{1}{\rho^2}\frac{2\mu E}{\beta^2\hbar^2}\right)u = 0$

Let $z \equiv \left(\frac{2\mu V_0}{\beta^2\hbar^2}\right)^{1/2}\rho$,

Then $\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{\nu^2}{z^2}\right)u = 0$, where $\nu^2 = \frac{E}{V_0}$

This is the Bessel equation of order ν.

Apply the boundary conditions:

BC #1: $u(r \to \infty) \to 0$

So $u(z=0) = 0$

Then $\boxed{u(z) = C\, j_\nu(z)}$, where C = const and j_ν = Bessel fn of order $\nu > 0$

BC#2: $u(r \to 0) \to 0$

So $u\left(z = \left(\frac{2\mu V_0}{\beta^2 \hbar^2}\right)^{1/2}\right) = 0$

Then $\boxed{u_n(z) = C J_{\nu_n}(z)}$ where ν_n is the n^{th} lowest positive number such that $J_{\nu_n}\left(\left(\frac{2\mu V_0}{\beta^2 \hbar^2}\right)^{1/2}\right) = 0$

The energy eigenvalues are given by

$\nu^2 = \frac{E}{V_0} \rightarrow \boxed{E_n = \nu_n^2 V_0}$

2 a $H\Psi = \left[\frac{-\hbar^2}{2m}\nabla^2 + V\right]\Psi = E\Psi$ (Let $V=0$ for simplicity since it is not specified in the problem)

$$\Psi = \frac{\pi}{2}^{-3/2} \sin(3x)\cdot \exp[i(5y+z)]$$

$$\nabla^2\Psi = \frac{d^2\Psi}{dx^2} + \frac{d^2\Psi}{dy^2} + \frac{d^2\Psi}{dz^2}$$

$$= \frac{\pi}{2}^{-3/2}(-9-25-1)\sin(3x)\exp[i(5y+z)]$$

So $\boxed{E = \frac{+\hbar^2}{2m}\cdot 35}$

b $p_{x_i}\Psi = -i\hbar\frac{d}{dx_i}\Psi$ $(i=1,2,3)$

$$\frac{d}{dx}\left[\frac{\pi}{2}^{-3/2}\sin(3x)\cdot\exp[i(5y+z)]\right] = 3\,\frac{\pi}{2}^{-3/2}\cos(3x)\exp[i(5y+z)]$$

So $p_x\Psi = -i\hbar 3\cot(3x)\cdot\Psi$

$$\frac{d}{dy}\left[\frac{\pi}{2}^{-3/2}\sin(3x)\exp[i(5y+z)]\right] = i5\Psi$$

So $p_y\Psi = -i\hbar\cdot i5\Psi = 5\hbar\Psi$

$$\frac{d}{dz}\left[\frac{\pi}{2}^{3/2}\sin(3x)\exp[i(5y+z)]\right] = i\Psi$$

So $p_z\Psi = -i\hbar\cdot i\Psi = \hbar\Psi$

(c) $\Psi(\vec{r},t) = \Psi(r,0)\cdot e^{-iEt/\hbar}$

$$= \frac{\pi^{-3/2}}{2}\sin(3x)\exp[i(5y+z)]\, e^{-\frac{i35\hbar t}{2m}}$$

(d) $p_x(t=0) = 3\hbar \rightarrow -i\hbar\frac{d}{dx}\Psi(x) = 3\hbar\Psi(x) \rightarrow \Psi(x) = e^{i3x}$

$p_y(t=0) = 5\hbar \rightarrow -i\hbar\frac{d}{dy}\Psi(y) = 5\hbar\Psi(y) \rightarrow \Psi(y) = e^{i5y}$

$p_z(t=0) = 1\hbar \rightarrow -i\hbar\frac{d}{dz}\Psi(z) = \hbar\Psi(z) \rightarrow \Psi(z) = e^{iz}$

$\Psi(x,y,z,t=0) = e^{i(3x+5y+z)}$ (unnormalized)

To find $\Psi(t\neq 0)$, need E.

$$-\frac{\hbar^2}{2m}\nabla^2\Psi = E\Psi$$

$$-\frac{\hbar^2}{2m}\left[\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right]e^{i(3x+5y+z)} = -\frac{\hbar^2}{2m}(-9-25-1)\,e^{i(3x+5y+z)}$$

$$= +\frac{35\hbar^2}{2m}\,e^{i(3x+5y+z)}$$

So $\Psi(\vec{r},t) = e^{i(3x+5y+z-\frac{35\hbar t}{2m})}$ (unnormalized)

(3) $$\nabla^2\psi + \underbrace{\left[\left(\frac{E}{\hbar c} + \frac{Ze^2}{\hbar c r}\right)^2 - \left(\frac{mc}{\hbar}\right)^2\right]}_{\text{call this } \rho(r)}\psi = 0$$

Plug in $\nabla^2 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - \frac{L^2}{\hbar^2 r^2}$:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\psi - \frac{L^2}{\hbar^2 r^2}\psi + \rho(r)\psi = 0$$

Postulate that $\psi = R(r)Y_{lm}(\theta,\phi)$

$$Y\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)R - \frac{R}{\hbar^2 r^2}L^2 Y + \rho(r)RY = 0$$

Multiply by $\frac{r^2}{RY}$:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)R - \frac{1}{\hbar^2 Y}L^2 Y + r^2\rho(r) = 0$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)R + r^2\rho(r) = \frac{1}{\hbar^2 Y}L^2 Y$$

Both sides must = the same constant C

RHS: So $\frac{1}{\hbar^2 Y}L^2 Y = C$

$$\downarrow$$

$$L^2 Y = C\hbar^2 Y$$

$$\downarrow$$

$$C = l(l+1)$$

Plug this into the LHS:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)R + r^2\rho(r) = \ell(\ell+1)$$

$$\boxed{\frac{d}{dr}\left(r^2\frac{d}{dr}\right)R + r^2\left[\left(\frac{E}{\hbar c}+\frac{Ze^2}{\hbar c r}\right)^2 - \left(\frac{mc}{\hbar}\right)^2\right]R - \ell(\ell+1)R = 0}$$

4) $$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

In ~~spherical~~ cylindrical coordinates,

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

Consider the case where $V(\rho) = \begin{cases} 0 & \text{for } \rho < a \\ \infty & \text{for } \rho \geq a \end{cases}$

(Let's ignore the z boundary conditions.)

Then

$$-\frac{\hbar^2}{2m}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right]\psi = E\psi \quad \text{for } \rho < a$$

Guess that $\psi = R(\rho)\Phi(\phi)$ (no z-dependence)

$$\Phi\frac{d^2R}{d\rho^2} + \Phi\frac{1}{\rho}\frac{dR}{d\rho} + R\frac{1}{\rho^2}\frac{d^2\Phi}{d\phi^2} = -\frac{2mE}{\hbar^2}R\Phi$$

multiply by $\rho^2/R\Phi$

$$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{1}{R}\rho\frac{dR}{d\rho} + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + \frac{2mE\rho^2}{\hbar^2} = 0$$

$$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} + \frac{2mE\rho^2}{\hbar^2} = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$

Both sides must equal the same constant "n^2"

RHS: $$-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = n^2$$

$\downarrow$

$$\frac{d^2\Phi}{d\varphi^2} + n^2\Phi = 0$$

$$\Phi = A\sin n\varphi + B\cos n\varphi$$

LHS: $\frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} + \frac{2mE}{\hbar^2}\rho^2 = n^2$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(\frac{2mE}{\hbar^2} - \frac{n^2}{\rho^2}\right)R = 0$$

*This is Bessel's Equation.
This equation may not be familiar to everyone in Physics 492, so students who got to this point get full credit. For students familiar with the Bessel equation, the solution is

$$R = CJ_n(\rho) + DN_n(\rho)$$

Set $D = 0$ to guarantee that R is regular at $\rho = 0$.

Then $\Psi = R\Phi$, normalize and insist that $\Phi(0) = \Phi(2\pi)$.

(5) Recall that for any Hermitian operator Q,

$$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle [H,Q]\rangle + \left\langle \frac{dQ}{dt}\right\rangle.$$

So if $Q \neq Q(t)$, Q is conserved if $[H,Q]=0$.

In this problem, $H = \frac{p_x^2}{2m} + V_0 \sin\left(\frac{2\pi x}{a}\right)$

and $Q = p_x$

$$[H,Q] = \frac{1}{2m}\underbrace{[p_x^2, p_x]}_{0} + V_0 \underbrace{\left[\sin\frac{2\pi x}{a}, p_x\right]}_{[f(x),\, p_x]}.$$

Recall that we showed in class that $[f(x), p] = i\hbar \frac{df}{dx}$

$$\text{So } [H,Q] = V_0 i\hbar \frac{d}{dx}\left(\sin\frac{2\pi x}{a}\right)$$

$$= V_0 i\hbar \frac{2\pi}{a}\cos\frac{2\pi x}{a} \neq 0$$

So p_x is not conserved.