Quantum Mechanics 2 Homework #10

1) Goswami problem 18.5.

2) Goswami problem 18.6.

3) Goswami problem 18.7.

4) Goswami problem 18.8.

5) Goswami problem 22.2.

QM 2

Answers to homework 10

(1) $$H_1 = \begin{cases} \dfrac{\hbar^2}{40ma^2} & \text{if } \dfrac{a}{2} < x < a \\ 0 & \text{otherwise} \end{cases}$$

The unperturbed eigenfunctions are given in Goswami Eq 1.19:

$$\varphi_n(x) = \begin{cases} \sqrt{\dfrac{2}{a}} \sin\left(\dfrac{n\pi x}{a}\right) & 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

These energies are $E_n = \dfrac{n^2\pi^2\hbar^2}{2ma^2}$, $n = 1, 2, 3, \ldots$

(a) $$E_n^{(1)} = \langle \varphi_n | H_1 | \varphi_n \rangle$$

$$= \int_{-\infty}^{+\infty} \varphi_n^*(x) H_1 \varphi_n(x)\, dx$$

$$= \int_{a/2}^{a} \left[\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)\right] \frac{\hbar^2}{40ma^2} \left[\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)\right] dx = \boxed{\frac{\hbar^2}{80ma^2}}$$

(b) $$\langle x | \psi_1 \rangle = |\varphi_1\rangle + \sum_{k \neq 1}' \frac{\langle \varphi_k | H_1 | \varphi_1 \rangle}{E_1^{(0)} - E_k^{(0)}} |\varphi_k\rangle$$

It turns out that $\langle \varphi_k | H_1 | \varphi_1 \rangle = 0$ for k odd.

To find the k = even terms,

$$\langle \varphi_{2\ell} | H_1 | \varphi_1 \rangle = \frac{\hbar^2}{20ma^3} \int_{a/2}^{a} \sin\left(\frac{2\ell\pi x}{a}\right) \sin\left(\frac{\pi x}{a}\right) dx$$

$$= \frac{(-1)^\ell \hbar^2}{10\pi ma^2} \frac{\ell}{(4\ell^2 - 1)}$$

②

The first three non-zero matrix elements therefore are:

$$\langle \varphi_2 | H_1 | \varphi_1 \rangle = \frac{-\hbar^2}{30\pi m a^2}$$

$$\langle \varphi_4 | H_1 | \varphi_1 \rangle = \frac{\hbar^2}{75\pi m a^2}$$

$$\langle \varphi_6 | H_1 | \varphi_1 \rangle = \frac{-3\hbar^2}{350\pi m a^2}$$

So $$\langle x | \psi \rangle = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) + \sum_{l=1}^{\infty} \frac{(-1)^l \hbar^2}{10\pi m a^2} \frac{l}{(4l^2-1)} \left[\frac{\pi^2\hbar^2}{2ma^2}\left(1-(2l)^2\right)\right]^{-1} \cdot \sqrt{\frac{2}{a}} \sin\left(\frac{2l\pi x}{a}\right) + \ldots$$

$$\boxed{= \sqrt{\frac{2}{a}}\left\{ \sin\frac{\pi x}{a} + \frac{1}{45\pi^3} \sin\frac{2\pi}{a} - \frac{2}{1125\pi^3} \sin\frac{4\pi}{a} + \frac{3}{6125\pi^3} \sin\frac{6\pi}{a} + \ldots \right\}}$$

(c) $$E_1^{(2)} = \sum_{l=1}^{\infty} \left[\frac{(-1)^l \hbar^2}{10\pi m a^2} \frac{l}{4l^2-1}\right]^2 \left[\frac{\pi^2\hbar^2}{2ma^2}\left[1-(2l)^2\right]\right]^{-1} \boxed{\approx -7.89\times 10^{-6} \frac{\hbar^2}{ma^2}}$$

2 Begin with Goswami Eq 18.13. Multiply by $\langle \varphi_m |$ $(m \neq n)$ from the left to get:

$$E_m^{(0)} c_{nm}^{(2)} + \sum_{k \neq n} c_{nk}^{(1)} \langle \varphi_m | H_1 | \varphi_k \rangle = E_n^{(0)} c_{nm}^{(2)} + E_n^{(1)} c_{nm}^{(1)}$$

Solve for $c_{nm}^{(2)}$:

$$\boxed{c_{nm}^{(2)} = \frac{1}{E_n^{(0)} - E_m^{(0)}} \left\{ \sum_{k \neq n} \frac{\langle \varphi_m | H_1 | \varphi_k \rangle \langle \varphi_k | H_1 | \varphi_n \rangle}{E_n^{(0)} - E_k^{(0)}} - \frac{\langle \varphi_m | H_1 | \varphi_n \rangle \langle \varphi_n | H_1 | \varphi_n \rangle}{E_n^{(0)} - E_m^{(0)}} \right\}}$$

where the substitutions $E_n^{(1)} = \langle \varphi_n | H_1 | \varphi_n \rangle$ and $c_{nk}^{(1)} = \dfrac{\langle \varphi_k | H_1 | \varphi_n \rangle}{E_n^{(0)} - E_k^{(0)}}$

are used.

③ In the $|\varphi_1\rangle, |\varphi_2\rangle, |\varphi_3\rangle$ basis,

$$H = H_0 + H_1 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix}$$

We will calculate to first order only.
Use non-degenerate perturbation theory for state $|\varphi_3\rangle$:

$$E_3^{(0)} = 4$$

$$E_3^{(1)} = \langle \varphi_3 | H_1 | \varphi_3 \rangle = 0$$

$$E_3^{(2)} = \sum_{k \neq n} \frac{|\langle \varphi_k | H_1 | \varphi_n \rangle|^2}{E_n^{(0)} - E_k^{(0)}} = \frac{|1|^2}{4-2} + \frac{|0|^2}{4-2} = \frac{1}{2}$$

So to first order, $E_3 = 4$

Use degenerate perturbation theory for states $|\varphi_1\rangle$ and $|\varphi_2\rangle$.
Diagonalize $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$:

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\lambda = 1, 3$$

Eigenfunctions:

when $\lambda = 1$, $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

when $\lambda = 3$, $\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

So the new states are: with energies to <u>first</u> order:

$$\text{"}|I\rangle\text{"} \equiv \frac{1}{\sqrt{2}}\left(|\varphi_1\rangle - |\varphi_2\rangle\right), \quad E_I = 0 + 1 = 1$$

$$\text{"}|II\rangle\text{"} \equiv \frac{1}{\sqrt{2}}\left(|\varphi_1\rangle + |\varphi_2\rangle\right) \quad E_{II} = 0 + 3 = 3$$

(4) $H = AS_z^2 + B(S_x^2 - S_y^2)$

$$S_x^2 - S_y^2 = \frac{1}{2}\left\{(S_x + iS_y)^2 + (S_x - iS_y)^2\right\}$$

$$= \frac{1}{2}S_+^2 + \frac{1}{2}S_-^2$$

The possible m states in a spin-1 system are $|+1\rangle, |0\rangle, |-1\rangle$

$$S_+|m\rangle = \hbar\sqrt{s(s+1) - m(m+1)}\,|m+1\rangle$$

$$S_-|m\rangle = \hbar\sqrt{s(s+1) - m(m-1)}\,|m-1\rangle$$

So $S_+^2|m\rangle = \hbar\sqrt{s(s+1) - m(m+1)}\,S_+|m+1\rangle$

$$= \hbar\sqrt{s(s+1) - m(m+1)} \cdot \hbar\sqrt{s(s+1) - (m+1)(m+2)}\,|m+2\rangle$$

etc for S_-^2

So $S_+^2|+1\rangle = 0$
$S_+^2|0\rangle = 0$
$S_+^2|-1\rangle = 2\hbar^2|+1\rangle$

$S_-^2|+1\rangle = 2\hbar^2|-1\rangle$
$S_-^2|0\rangle = 0$
$S_-^2|-1\rangle = 0$

Also $S_z|m\rangle = m\hbar|m\rangle$, so
$S_z^2|m\rangle = m^2\hbar^2|m\rangle$

$$H = AS_z^2 + \frac{B}{2}S_+^2 + \frac{B}{2}S_-^2 =$$

$$A\begin{pmatrix} \hbar^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hbar^2 \end{pmatrix} + \frac{B}{2}\begin{pmatrix} 0 & 0 & 2\hbar^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{B}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\hbar^2 & 0 & 0 \end{pmatrix}$$

$$= \hbar^2\begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}$$

The matrix is arranged:

$$\begin{array}{c|ccc} & |+1\rangle & |0\rangle & |-1\rangle \\ \hline \langle +1| & & & \\ \langle 0| & & & \\ \langle -1| & & & \end{array}$$

Note there is an incorrect factor of 2 here in the textbook

Diagonalize:

$$\begin{vmatrix} A\hbar^2-\lambda & 0 & B\hbar^2 \\ 0 & -\lambda & 0 \\ B\hbar^2 & 0 & A\hbar^2-\lambda \end{vmatrix} = 0$$

You get $\lambda_1 = (A+B)\hbar^2$
$\lambda_2 = 0$
$\lambda_3 = (A-B)\hbar^2$

(5) $P = |c_2(t)|^2$

$$c_2 = \delta_{20} - \frac{i}{\hbar}\int_0^t dt'\, e^{i(E_2-E_i)t'/\hbar} \langle u_2 | \delta\omega\, x^2 \cos ft | u_0 \rangle$$

Since $E_2 - E_0 = 2\hbar\omega$
and $\delta_{20} = 0$, we can rewrite this as

$$c_2 = \frac{-i}{\hbar}\int_0^t dt'\, e^{i2\omega t'} \delta\omega \cos ft \langle u_2 | x^2 | u_0 \rangle$$

$$= \frac{-i}{\hbar}\delta\omega \langle u_2 | x^2 | u_0 \rangle \frac{1}{2}\int_0^t dt' \left[e^{i(2\omega+f)t'} + e^{i(2\omega-f)t'}\right]$$

$$= \frac{-i}{\hbar}\delta\omega \langle u_2 | x^2 | u_0 \rangle \frac{1}{2}\left[\frac{e^{i(2\omega+f)t} - 1}{2\omega+f} + \frac{e^{i(2\omega-f)t} - 1}{2\omega-f}\right]$$

$$\langle u_2 | x^2 | u_0 \rangle = \langle u_2 | x | u_1 \rangle \langle u_1 | x | u_0 \rangle$$

$$= \sqrt{2}\sqrt{\frac{\hbar}{2m\omega}} \cdot \sqrt{1}\sqrt{\frac{\hbar}{2m\omega}} = \frac{\hbar}{\sqrt{2}\, m\omega}$$

So $c_2 = \frac{-i(\delta\omega)}{\sqrt{2}\, m\omega} e^{i\omega t}\left(e^{ift/2}\frac{\sin[(\omega+f/2)t]}{2\omega+f} + e^{-ift/2}\frac{\sin[(\omega-f/2)t]}{2\omega-f}\right)$

$$\boxed{P = |c_2|^2 = \frac{(\delta\omega)^2}{2m\omega^2}\left\{\frac{\sin^2\left[\left(\frac{2\omega+f}{2}\right)t\right]}{(2\omega+f)^2} + \frac{\sin\left[\left(\frac{2\omega-f}{2}\right)t\right]}{(2\omega-f)^2} + \frac{2\cos(ft)\sin\left[\left(\frac{2\omega+f}{2}\right)t\right]\sin\left[\left(\frac{2\omega-f}{2}\right)t\right]}{4\omega^2-f^2}\right\}}$$