Quantum Mechanics Homework #1

An accurate and complete diagram or the words "no diagram is possible" must accompany every solution.

1. Consider a particle which has the following wave function:

$\psi(x) = 0 \qquad x < -L/2$

$\psi(x) = Ae^{ikx}\cos(3\pi x/L) \qquad -L/2 \le x \le +L/2$

$\psi(x) = 0 \qquad x > +L/2$

(a) Normalize the wave function.

(b) Find the probability that the particle will be found in the region $0 \le x \le +L/4$.

2. No optical instrument can resolve the structural details of an object smaller than the wavelength of light by which it is being observed. For this reason, although an optical microscope using light of wavelength 5000 $\mathring{A}$ would be unable to observe a virus of diameter 200 $\mathring{A}$, an electron microscope would work. Calculate the voltage through which electrons must be accelerated to give them a de Broglie wavelength 1000 times smaller than the diameter of the virus. In this problem (unlike most problems in this course), you will need to use the relativistic expression for the total energy, $E = \sqrt{p^2c^2 + m^2c^4}$.

3. Devise a wave function that simultaneously satisfies the following properties of particles:

(a) $E = h\upsilon$

(b) $p = h/\lambda$

(c) $E = \sqrt{p^2c^2 + m^2c^4}$

Quantum Mechanics Homework #1, continued

4. Consider the wavefunction, $\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t}$.

(a) Normalize this wavefunction.

(b) Calculate the expectation value of x^2.

5. Consider a particle with de Broglie wavelength 10^{-8} cm traveling in a region of constant potential V_0. What is the wave function for that particle?

Quantum Mechanics Homework #2

1. Goswami problem 1.6.

2. Goswami problem 1.10.

3. Consider an unstable particle that has lifetime τ. The total probability of finding this particle somewhere is not 1, but rather $e^{-t/\tau}$, where τ is a constant. It is possible to describe such a situation by assuming that the potential V which controls the particle is complex.

(a) Write down a general complex potential, then show how the fact that it is complex modifies the probability continuity equation.

(b) Find a relationship between the complex V and τ.

4. Evaluate the following integrals:

(a) $\displaystyle\int_{-3}^{+1}\left(x^3-3x^2+2x-1\right)\delta\left(x+2\right)dx$

(b) $\displaystyle\int_{-1}^{+1}e^{|x|+3}\delta\left(x-2\right)dx$

Quantum Mechanics Homework #2, continued.

5. Demonstrate the following properties of the Dirac delta function:

(a) $x\delta(x-a) = a\delta(x-a)$

(b) $\delta(-x) = \delta(x)$

(c) $\delta(ax) = \frac{1}{|a|}\delta(x)$ for $a \neq 0$

(d) $\delta\left(x^2 - a^2\right) = \frac{1}{2|a|}\left[(x-a) + (x+a)\right]$ for $a \neq 0$

(e) $\int f(x)\delta'(x)\,dx = -f'(0)$ where the prime denotes differentiation

Quantum Mechanics Homework #3

1. Goswami problem 2.3

2. Consider a wavepacket with the following momentum distribution:

$$A(k,0)=\begin{cases}\dfrac{1}{\sqrt{\Delta k}} & \text{if } |k|<\dfrac{\Delta k}{2}\\[2ex] 0 & \text{if } |k|>\dfrac{\Delta k}{2}\end{cases}$$

(a) Find the particle's wavefunction $\psi(\mathrm{x})$.

(b) Use the precise definitions of Δp and Δx to calculate $\Delta p \cdot \Delta x$. Compare this product to the Uncertainty Principle.

3. (a) Show that with certain assumptions the group velocity is equal to dE/dp.

(b) What are the phase and group velocities of an electron whose de Broglie wavelength is 0.01 $\mathring{A}$?

4. Consider the spreading of the wavepacket of a free particle, for which $\omega = \hbar k^2/2m$.

(a) Find the fractional change in the size of the wavepacket if the particle is an electron with packet width 10^{-8} cm.

(b) Compare your answer in part (a) to the fractional change that would occur if the wavepacket has size 1 cm and represents an object of mass 1 gram.

5. Consider a set of wavefunctions $\psi_n = e^{-i2\pi nx/L}$. Show that wavefunctions belonging to different values of n are orthogonal.

Quantum Mechanics Homework #4

1. Consider an operator W that produces the following effect when applied to a wavefunction:

$$W\psi(x) = \psi(x+a)$$

The symbols in the parentheses indicate the functional dependence of ψ. Suppose that a is a very small number (think Taylor Series). Show that W may be expressed in terms of the momentum operator.

2. Consider a particle with wavefunction $\Psi(x,t) = A^{-a\left[\left(mx^2/\hbar\right)+it\right]}$. A and a are positive real constants. For what potential function V does Ψ satisfy the Schroedinger Equation?

3. Goswami problem 3.5.

4. Goswami problem 3.7.

5. Consider a particle whose initial state wavefunction is $\Psi(x,t=0) = Ae^{-ax^2}$.

(a) Normalize $\Psi(x,t=0)$.

(b) Find $\Psi(x,t)$.

(c) Plot $|\Psi|^2$ for $t=0$ and for two later times.

(d) Calculate the uncertainty product, $\Delta x \cdot \Delta p$, for this wave.

Quantum Mechanics Homework #5

1. Calculate the following commutators.

(a) $[x, p_x]$

(b) $[x, p_y]$

(c) $[p_x, p_y]$

Do your results depend upon whether you represent the operators in coordinate space or momentum space?

2. Show that if the first spatial derivative of the wavefunction is discontinuous across a boundary, then the time-independent Schroedinger Equation predicts (non-physical) infinite energy, infinite potential, infinite mass, or infinite probability current.

3. Find the momentum space representation of the position operator x.

4. Goswami problem 3.9.

5. A particle of total energy $9V_0$ is incident from the left upon a potential given by

$$V = 8V_0 \qquad \text{for } x < 0$$
$$V = 0 \qquad \text{for } 0 \le x < a$$
$$V = 5V_0 \qquad \text{for } x > a$$

Find the probability that the particle will be transmitted to the right of $x = a$.

Quantum Mechanics Homework #6

1. Find the minimum depth V_0, in electron volts, required for a square well to contain two allowed energy levels, if the width of the well is $2a = 4\times 10^{-13}$ cm and the particle has mass 2 GeV/c^2.

2. Find the commutator of the parity operator and the kinetic energy operator.

3. Derive the expression for the transmission coefficient, "T^2" (Goswami equation 4.13). Then find the reflection coefficient, "R^2". Use these to show conservation of probability.

4. The potential for a particle of mass m moving in one dimension is

$$V(x) = \begin{cases} \infty & \text{for } x < 0 \\ 0 & \text{for } 0 \le x < L \\ V_0 & \text{for } L \le x < 2L \\ \infty & \text{for } x \ge 2L \end{cases}$$

Assume that the energy of the particle is in the range $0 < E < V_0$. Find the energy eigenfunctions and the equation that determines the energy eigenvalues. You do not have to solve for the energy eigenvalues explicitly. You also do not have to normalize the eigenfunctions.

5. Calculate the eigenfunctions and eigenvalues of a particle of mass m in an infinite square well of width a.

Quantum Mechanics Homework #7

1. A particle of mass m is within an infinite square well potential of width L. At time $t = 0$, its wavefunction is

$$\Psi(x,0) = \frac{1}{\sqrt{3}}\sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L} + \sqrt{\frac{2}{3}}\sqrt{\frac{2}{L}}\cos\frac{4\pi x}{L}$$

(a) What will be the wavefunction at a later time $t = t'$?

(b) Is this a stationary state?

2. Goswami problem 4.13.

3. Goswami problem 4.14.

4. (a) Formalism Supplement problem 3.9 (page 86).

(b) Formalism Supplement problem 3.10 (page 86).

5. Formalism Supplement problem 3.22 (page 94).

Quantum Mechanics Homework #8

Remember to include a figure with each problem for which a figure is possible.

1. Consider the Class 2 (sine function) solution of the finite square well.

(a) Carry out the graphical solution for the allowed energies of these states.

(b) What condition must hold in order for there to be at least one bound Class 2 solution?

2. Consider the operator xp.

(a) Show that it is not Hermitian.

(b) Construct an operator that corresponds to this physical observable (that is, reflects the sequential measurement of position and momentum) but which is Hermitian.

3. Consider an operator W defined by $W = |\phi\rangle\langle\psi|$. Convert the following expressions from Dirac notation to calculus notation (that is, remove all the bras and kets).

(a) $\langle f|W$

(b) $W|f\rangle$

(c) $\langle f|W|\psi\rangle$

Quantum Mechanics Homework #9

1. Goswami problem 6.6.

2. Write the following expressions in Dirac notation.

(a) $f(x) = g(x)$

(b) $c = \int g^*(x')h(x')\,dx'$

(c) $f(x) = \sum_n \phi_n(x) \int \phi_n^*(x') f(x')\,dx'$

(d) $W = \psi(x) \int dx' \phi^*(x')$

(e) $\dfrac{\partial f(x)}{\partial x} = h(x) \int h^*(x') g(x')\,dx'$

3. Show that $\langle p | \psi \rangle = \psi(p)$.

Quantum Mechanics Homework #9, continued

4. Consider the wavefunction

$$\Psi(x,0) = \frac{1}{\sqrt{2}}\left[\phi_1(x) + \phi_2(x)\right],$$

where $\phi_1(x)$ and $\phi_2(x)$ are two orthonormal stationary states of some arbitrary potential.

(a) Calculate the time necessary for $\Psi(x,t)$ to evolve into a state orthogonal to $\Psi(x,0)$. Call your answer "Δt".

(b) Define "ΔE" as the energy difference between the states $\phi_1(x)$ and $\phi_2(x)$. Show that $\Delta E \cdot \Delta t$ satisfies the Uncertainty Principle.

5. Use the Generalized Uncertainty Relation and identities you know concerning commutator manipulations to find $\Delta x \cdot \Delta E$ in terms of p_x and constants.

Quantum Mechanics Homework #10

1. Goswami problem 7.1.

2. Goswami problem 7.4.

3. Consider a particle which is known to be in the ground state of the harmonic oscillator. What is the probability of finding the particle outside the classically allowed region?

4. For a function $f(x)$ that can be expanded in a Taylor Series, show that $f(x+x_0) = e^{ipx_0/\hbar} f(x)$, where x_0 is a distance. Here the operator $p/\hbar$ generates translations in space just as $-H/\hbar$ generates translations in time.

5. Consider the relationship between the time derivative of the expectation value of an operator Q and the commutator of Q with the Hamiltonian.

(a) Write down the relationship.

Consider separately the cases where Q is

(b) 1.

(c) H, the same Hamiltonian.

Show that in each case the relationship leads to a conservation law. State what is conserved.

Quantum Mechanics Homework #11

1. Goswami problem 7.10, parts a, b, and c only.

2. Goswami problem 7.12.

3. List 3 equivalent formulas that you have learned for the Hermite functions. Then calculate $H_6(\xi)$ from any one of them.

4. Consider a harmonic oscillator constructed from a 1 gram mass at the end of a spring. The oscillator frequency is 1 Hz and the mass passes through its equilibrium position with a frequency of 10 cm/s. What is the quantum number of this energy state?

5. Convert the following expressions into Dirac Notation:

(a) $\psi(p)$

(b) $\psi^*_{E_n}(x)$

(c) $\dfrac{e^{-ip'x/\hbar}}{\sqrt{2\pi\hbar}}$

(d) $\delta(x-x')$

Convert the following expressions out of Dirac Notation:

(e) $\int dx' \langle x|x'\rangle\langle x'|\psi\rangle$

(f) $\langle p|\ell\rangle\langle\ell|g\rangle = \dfrac{\partial\langle p|f\rangle}{\partial p}$

(g) $\langle p|f\rangle = \sum_j \langle p|\phi_j\rangle\langle\phi_j|f\rangle$

(h) $\int |x\rangle dx \langle x|$

Quantum Mechanics Homework #12

1. Construct the matrix for the momentum operator in the basis $|u_n\rangle$.

2. Goswami problem 7.A6.

3. Goswami problem 7.A7.

4. Write out exactly (that is, including normalization) the wavefunctions for the three lowest energy states of the harmonic oscillator. Write everything in terms of x, not ξ.

5. Prove that diagonal matrices always commute. It follows from this that simultaneously diagonalizable matrices---that is, operators with simultaneous eigenstates---must commute.

Quantum Mechanics Homework #13

1. Goswami problem 8.1.

2. Goswami problem 8.3.

3. Consider a particle in a harmonic oscillator potential. The particle has initial wave function

$$\Psi(x,0) = A\left[\phi_0(x) + \phi_1(x)\right]$$

where A is a constant and the ϕ_i are stationary states of the harmonic oscillator potential.

(a) Normalize $\Psi(x,0)$.

(b) Find $\Psi(x,t)$ and $|\Psi(x,t)|^2$.

(c) Find the expectation value of x as a function of time. The expectation value oscillates sinusoidally. State what its amplitude and angular frequency are.

(d) Find the expectation value of p.

(e) Demonstrate that Ehrenfest's Second Equation is value for this system.

Quantum Mechanics Homework #14

1. Use the WKB Approximation to find the allowed energies of the harmonic oscillator.

2. Use the WKB Approximation to find the first 4 eigenfunctions of the harmonic oscillator. Plot the WKB eigenfunctions and the exact eigenfunctions for the first 4 levels and compare them.

3. Goswami problem 9.9.

4. Two particles of masses m_1 and m_2, respectively, in a one-dimensional infinite well of width L, are known to be in a state given by

$$\psi\left(x_1, x_2\right) = \frac{\left[3\psi_5(x_1)\psi_4(x_2) + 7\psi_9(x_1)\psi_8(x_2)\right]}{\sqrt{58}}$$

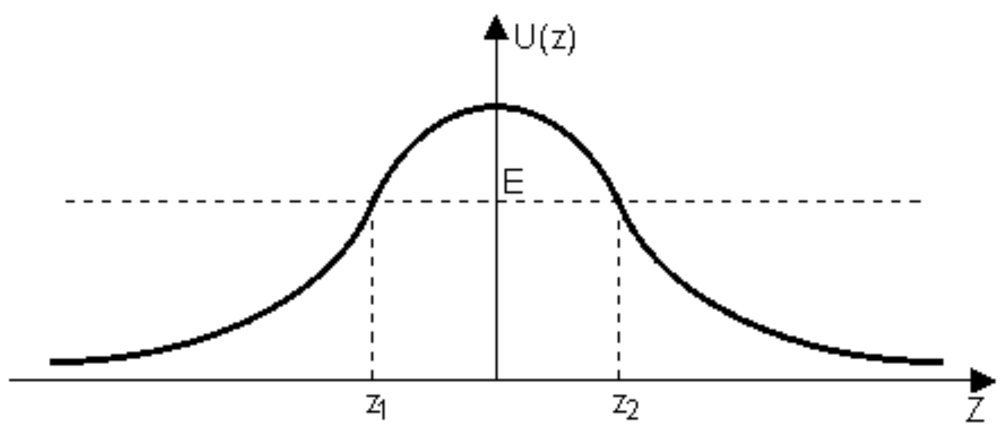
If the energy of the system is measured, what values will be found and with what probabilities will they occur?

5. Develop appropriate connection formulas to analyse the problem of scattering from a barrier with sloping walls (pictured below). Begin by writing the WKB wave function in the form below. Calculate the tunneling probability $|F|^2/|A|^2$. Do not assume that the barrier is symmetric.

$$\frac{1}{\sqrt{p(z)}}\left[A\exp\left(\frac{i}{\hbar}\int_z^{z_1} p(z')dz'\right)+B\exp\left(\frac{-i}{\hbar}\int_z^{z_1} p(z')dz'\right)\right] \qquad z<z_1$$

$$\frac{1}{\sqrt{P(z)}}\left[C\exp\left(\frac{1}{\hbar}\int_{z_1}^{z} P(z')dz'\right)+D\exp\left(\frac{-1}{\hbar}\int_{z_1}^{z} P(z')dz'\right)\right] \qquad z_1<z<z_2$$

$$\frac{1}{\sqrt{p(z)}}\left[F\exp\left(\frac{i}{\hbar}\int_{z_2}^{z} p(z')dz'\right)\right] \qquad z<z_1$$



Quantum Mechanics Homework #15

1. Goswami problem 9.1.

2. Goswami problem 9.6.

3. Consider two non-interacting particles in an infinite square well of width L. Both have mass m.

(a) Suppose that the particles are indistinguishable fermions. Write the wavefunction for the first excited state above groundm and calculate the energy of this state.

(b) Suppose that the particles are indistinguishable bosons. Write the wavefunction for the first excited state above ground, and calculate the energy of this state.

(c) Suppose that the particles are somehow distinguishable but have the same mass. Write the wavefunction for the first excited state above ground, and calculate the energy of this state.

Quantum Mechanics Homework #15, continued

4. Consider a system of particles that are indistinguishable but for the purposes of constructing wavefunctions can be numbered from 1 to N. These particles are simultaneously confined in some potential. Each of them could be in any energy state from the selection {a, b, c, ... n}. If any one of these particles were in isolation, its wavefunction would be denoted by one of the following symbols:

$$\psi_a(1),\psi_b(1),...,\psi_n(1),\psi_a(2),\psi_b(2),...,\psi_n(2),...\psi_a(N),\psi_b(N),...,\psi_n(N),$$

where the argument indicates the particle number and the subscript indicates the state.

(a) Construct the most general symmetric wavefunction for the combined state of these particles.

(b) Construct the most general antisymmetric wavefunction for the combined state of these particles.

Quantum Mechanics Homework #16

1. Goswami problem 11.1.

2. Goswami problem 11.3.

3. Goswami problem 11.4.

4. Show that, for a particle in a potential $V(\vec{r})$, the rate of change of the expectation value of $\vec{L}$ is equal to the expectation value of the torque:

$$\frac{d\left\langle \vec{L} \right\rangle}{dt} = \left\langle \vec{N} \right\rangle,$$

where $\vec{N} = \vec{r} \times \left(-\vec{\nabla} V\right)$. This is the rotational analog of Ehrenfest's second equation.

5. Calculate $Y(\ell = 1, m = 3)$ directly from the definition of the spherical harmonics.

Quantum Mechanics Homework #17

1. Write out the explicit formulas for $Y(\ell=1,m=1)$ and $Y(\ell=1,m=2)$. Write the formula for the angular momentum raising operator in spherical coordinates. Apply the raising operator to $Y(\ell=1,m=1)$ directly to get $Y(\ell=1,m=2)$.

2. Goswami problem 11.5.

3. The Hamiltonian for a certain rotating object is given by

$$H=\frac{L_x^2+L_y^2}{2I_1}+\frac{L_z^2}{2I_2}$$

What are the eigenvalues of H? Draw the spectrum for the case where $I_1 > I_2$.

4. Goswami problem 11.7. Use raising and lowering operators to do this.

5. Goswami problem 11.A6.