Physics 301

Homework due 5 October 2022

1) Stowe problem 4-2.

2) Consider a Gaussian distribution, $P(x) = Ae^{-Bx^2}$. Use the normalization of probability to show that

$$A = \sqrt{\frac{B}{\pi}}.$$

3) Stowe problem 4-5.

4) Stowe problem 4-6.

5) We showed in class that the Gaussian distribution is an approximation to the binomial distribution for the case where N is very large. The following problem is intended to help you derive the Poisson distribution, which is an approximation to the binomial distribution when $n << N$ and $p << 1$. Begin with the binomial distribution: $P_N(n) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$.

(a) Use the fact that $\ln(1 - p) \approx -p$ to show that $(1-p)^{N-n} \approx e^{-Np}$

(b) Show that $\frac{N!}{(N-n)!} \approx N^n$

(c) Apply the results above to the binomial distribution to show that $P_N(n) \approx \frac{\lambda^n e^{-\lambda}}{n!}$ $(n << N \text{ and } p << 1)$ where $\lambda = Np$ is the mean number of events. The equation in the line above describes the Poisson distribution.

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1. 10^8 dice $= N$
\# with 1 dot up

$p = \frac{1}{6}$; $q = \frac{5}{6}$.

a) $\bar{n} = pN = \frac{1}{6} \cdot 10^8 = 1.7 \times 10^7$

b) $\sigma = \sqrt{Npq} = \sqrt{10^8 \cdot \frac{1}{6} \cdot \frac{5}{6}} = \frac{\sqrt{5}}{6} \times 10^4 = 3.7 \times 10^3$

c) $\frac{\sigma}{\bar{n}} = \frac{3.7 \times 10^3}{1.7 \times 10^7} = 2.2 \times 10^{-4}$

2) To show that $A = \left(\frac{B}{\pi}\right)^{1/2}$ if $P(x) = Ae^{-Bx^2}$

$$\int P(x)\,dx = 1$$

$$\int P(x)\,dx \int P(y)\,dy = 1^2 = 1$$

$$\int P(x)P(y)\,dx\,dy = 0 \qquad r^2 = x^2 + y^2$$

$$dx^2\,dy^2 = r\,dr\,d\theta$$

$$A^2 \int e^{-B(x^2+y^2)}\,dx\,dy = 1$$

$$A^2 \underbrace{\int e^{-Br^2}\,r\,dr}_{\frac{\Gamma(1)}{2B}}\underbrace{d\theta}_{2\pi} = 1$$

$$\int_0^\infty x^n e^{-ax^p}\,dx = \frac{\Gamma(k)}{p a^k} \qquad k = \frac{n+1}{p}$$

Here $n = 1$

$a = B$

$p = 2$

So $k = 1$

$\Gamma(1) = 1$

Thus

$$A^2 \cdot \frac{1}{\cancel{2}B} \cdot \cancel{2}\pi = 1$$

$$A^2 = \frac{B}{\pi}$$

$$A = \sqrt{\frac{B}{\pi}}$$

3) 600 dice $= N$

with 6 dots up

$p = \frac{1}{6}$

a) $\bar{n} = pN = \frac{1}{6} \cdot 600 = 100$

b) $\sigma = \sqrt{Npq} = \sqrt{600 \cdot \frac{1}{6} \cdot \frac{5}{6}} = \frac{\sqrt{3000}}{6} = 9.13$

c) $P_{600}(n) = A e^{-B(n-\bar{n})^2}$

$A = \frac{1}{\sqrt{2\pi}\sigma} = \frac{1}{\sqrt{2\pi} \cdot 9.13} = 0.043$

$B = \frac{1}{2\sigma^2} = \frac{1}{2(9.13)^2} = 0.006$

d) P (exactly 100) =

$n = 100$

$\frac{(n-\bar{n})^2}{2\sigma^2} = \frac{(100-100)^2}{2(9.13)^2} = 0$

$\frac{1}{\sqrt{2\pi}\sigma} = 0.043$

$P_{600}(100) = 0.043 e^{-0} = 0.043$

e) $\frac{(n-\bar{n})^2}{2\sigma^2} = \frac{(96-100)^2}{2(9.13)^2} = 0.096$

$P_{400}(96) = 0.043\,e^{-0.096} = 0.039$

4) $N = 600$

$n = 100$

$$\frac{N!}{n!(N-n)!} = \frac{600!}{100!\,500!} = \frac{1.2\times10^{1408}}{(9.3\times10^{157})(1.2\times10^{1134})}$$

$$= 0.1\times10^{117} = 10^{116}$$

5 a) Begin with $\ln(1-p)$. Expand it in a Taylor Series:

$\ln(1-p) \approx -p - \frac{1}{2}p^2 - \ldots$ For small p this is $\approx -p$

Consider $\ln\left[(1-p)^{N-n}\right]$. Expand it using properties of li

$$\downarrow$$

$$(N-n)\ln(1-p)$$

$$\downarrow$$

$\approx (N-n)(-p)$. Take the exponential

So $\exp\left[\ln(1-p)^{N-n}\right] = (1-p)^{N-n} \approx \exp\left[(N-n)(-p)\right]$

Consider the case were n is small and $\ll N$. So

$\exp(np - Np) \approx \boxed{e^{-Np}}$

b) $$\frac{N!}{(N-n)!} = \frac{N \cdot (N-1) \cdot (N-2) \cdot \ldots \cdot (2) \cdot (1)}{(N-n) \cdot (N-n-1) \cdot (N-n-2) \cdot \ldots \cdot (2) \cdot (1)}$$

$$= \underbrace{N \cdot (N-1) \cdot (N-2) \cdot \ldots \cdot (N-n+1)}$$

Notice that this expression has n factors

when $n \ll N$, each factor is $\approx N$.

So the RHS $\boxed{\approx N^n}$

c) Begin with

$P_N(n) = \frac{N!}{n!(N-n)!} p^n \cdot (1-p)^{N-n}$. Make the above substitutions:

$$P_N(n) \approx \frac{1}{n!} \cdot N^n \cdot p^n \cdot e^{-Np} = \frac{1}{n!}(Np)^n e^{-Np}$$

Define $\lambda = Np$, Then

$$P_N(n) \approx \boxed{\frac{1}{n!}\lambda^n e^{-\lambda}}$$