Physics 301

Homework due 14 September 2022

1) Find an expression for the density of states of a relativistic, massless, classical particle confined to the range $0 < r < r_0$ and $0 < p < p_0$.

2) Stowe problem 3-8.

3) Consider a system of 100 air molecules in an otherwise empty room. Calculate the probability that 49 are in the front 60% of the room and 55 are in the top half.

4) Give an example of a process that involves continuous change but is

(a) quasi-static

(b) not quasi-static

Support both answers numerically. Choose examples that are different from those in the Stowe book.

5) How would the H-Theorem be changed for a system in which the transition rate W_{rs} between states r and s were not equal to the rate for the reverse process?

PHYS 10 501

Answers to homework due 14 Sept. 2022

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(1) General expression for the

$$\# \text{ quantum state in a finite volume} = \int_r \int_p \frac{1}{h^3} \, dV_r \, dV_p$$

$$= \int_p \frac{4\pi V_r}{h^3} p^2 \, dp$$

Let $E = pc$, so

$$p^2 = \frac{E^2}{c^2} \text{ and}$$

$$dp = \frac{1}{c} dE$$

$$= \int_0^{cp_0} \underbrace{\frac{4\pi V_r}{c^3 h^3} E^2}_{g(E)} dE$$

2) a) $P_N(n) = \frac{N!}{n!(N-n)!} p^n q^{N-n}$

Here p = prob of 4 dots up $= \frac{1}{6}$

q = prob of not 4 dots up $= \frac{5}{6}$

N = # elements (dice) = 8

n = # of elements satisfying criterion "p" = 5

Plug in

$$P_8(5) = \frac{8!}{5!\,3!}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^3 = \boxed{0.00417}$$

b) The number of configurations is given by $\frac{N!}{n!(N-n)!}$

Here the number is $\frac{8!}{5!\,3!} = \boxed{56}$

c) Again use $P_N(n) = \frac{N!}{n!(N-n)!} p^n q^{N-n}$

but now add the cases for $n = 5, 6, 7, 8$

$$P_{TOT} = \sum_{n=5}^{8} \frac{8!}{n!(8-n)!}\left(\frac{1}{6}\right)^n\left(\frac{5}{6}\right)^{8-n} = 0.00417$$

+ 0.00042

+ 0.00002

0.

$\boxed{0.00461}$

3. Using Stowe equation 3.14,

$P_{100}(49,55) = P_{100}(49) \cdot P'_{100}(55)$.

Each term is separately calculated with equation 3.12,

$P_N(n) = \frac{N!}{n!(N-n)!} p^n q^{N-n}$. Plugging in,

$$P_{100}(49,55) = \frac{100!}{49!\,51!}(0.6)^{49}(0.4)^{51} \times \frac{100!}{55!\,45!}(0.5)^{55}(0.5)^{45}$$

To find the factorials, use Stirling's approximation,

$\ln(m!) \approx m \ln m - m + \frac{1}{2}\ln(2\pi m)$

So $\ln(100!) = 100 \ln 100 - 100 + \frac{1}{2}\ln(2 \cdot \pi \cdot 100) = 363.6$

So $100! \approx e^{+364}$

Similarly, $49! \approx e^{144}$

$51! \approx e^{152}$

$55! \approx e^{168}$

$45! \approx e^{129}$

Also $(0.6)^{49} = 1.35 \times 10^{-11}$

$(0.4)^{51} = 5.07 \times 10^{-21}$

$(0.5)^{100} = 7.89 \times 10^{-31}$

So $P_{100}(49,55) = e^{(364+364-144-152-168-129)} \times 1.35 \times 5.07 \times 7.89 \times 10^{-63}$

$= \boxed{0.0023}$

4) A process is quasi-static if it occurs over a time period that is long compared to the time it takes the system to return to equilibrium after being perturbed by the process. Some order-of-magnitude relaxation time τ are:

- for quantum mechanical transitions, $\Delta t \cdot \Delta E \approx \hbar$, so

$$\tau \approx \frac{\hbar}{\Delta E} = \frac{10^{-15}\ eV/sec}{10\ eV} \approx 10^{-16}\ sec$$

For air in a room: order of magnitude estimates:

10^{28} molecules
Room volume $= 60\ m^3$
mean distance between molecules: 2×10^{-9} m
Room temperature: 300 K
molecule mass $\cong 3 \times 10^{-26}$ kg
Typical particle velocity $= V \cong \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3 \cdot (1.38 \times 10^{-23} \frac{J}{K})(30}{3 \times 1}}$

$= 643$ m/s

Suppose that a perturbation occurs in 1 corner of the room; we want to estimate how long it takes for the effect to propagate to the opposite corner.

Typical time for transport of the perturbation:

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt[3]{60 m^3}}{643\ m/s} = 4 \times 10^{-3} \text{ sec}$$

So if the air in a room is compressed during a time significantly shorter than 10^{-3} sec, the compression is not quasi-static; if the compression period is significantly longer than 10^{-3} sec, the process is quasi-static.

(6)

5.) We redo the H Theorem for the case where $W_{sr} \neq W_{rs}$. We follow the format used in the handout given to the class.

Consider P_r, the probability that a system is in state r. If any of the other states of the system can be labelled "s", then the change per unit time in the probability of being in state r is

$$\frac{dP_r}{dt} = \sum_s P_s W_{sr} - \sum_s P_r W_{rs} \quad \text{, where}$$

W_{rs} = transition rate out of r and

W_{sr} = " " " into r

Define $H \equiv \sum_r P_r \ln P_r$. Then

$$\frac{dH}{dt} = \sum_r \left(\frac{dP_r}{dt} \ln P_r + \frac{dP_r}{dt} \right) = \sum_r \frac{dP_r}{dt} (\ln P_r + 1)$$

"Eq 1" $$\frac{dH}{dt} = \sum_r \sum_s \left(P_s W_{sr} - P_r W_{rs} \right) (\ln P_r + 1)$$

Rewrite Eq 1 with the summation indices exchanged:

"Eq 2" $$\frac{dH}{dt} = \sum_s \sum_r \left(P_r W_{rs} - P_s W_{sr} \right) (\ln P_s + 1)$$

Add Eq 1 + Eq 2:

$$\frac{2\,dH}{dt} = \sum_r \sum_s{}' (P_s W_{sr} - P_r W_{rs})(\ln P_r - \ln P_s)$$

$$\downarrow$$

$$\frac{dH}{dt} = -\frac{1}{2}\sum_r \sum_s{}' (P_r W_{rs} - P_s W_{sr})(\ln P_r - \ln P_s)$$

In this case we can no longer be sure that the the argument of the sum is ≥ 0. The weaker condition we get so that $\frac{dH}{dt} \leq 0$, if:

$\ln P_r - \ln P_s \geq 0$ AND $(P_r W_{rs} - P_s W_{sr}) \geq 0$

$\downarrow$ $\qquad\qquad\qquad\qquad \downarrow$

$P_r > P_s$ $\qquad\qquad\qquad 1 < \frac{P_r}{P_s} > \frac{W_{sr}}{W_{rs}}$