Physics 301

Homework due 16 November 2022

1) Consider a gas of atoms, each having mass m, maintained at Kelvin temperature T inside an enclosure. The atoms emit light which passes (in the x-direction) through a window of the enclosure and can be observed as a spectral line. A stationary atom would emit light at the well defined frequency υ_0. Because of the Doppler Effect, the frequency of the light observed from the atom having an x-component of velocity v_x is not υ_0 but rather $\upsilon = \upsilon_0\left(1+\frac{v_x}{c}\right)$, where c is the speed of light. As a result of the Doppler Effect, the light registered by the spectroscope has a distribution of frequencies.

(a) Calculate the mean frequency $\bar{\upsilon}$ of the light observed; (b) Calculate the dispersion, $\overline{(\Delta\upsilon)^2} \equiv \overline{(\upsilon-\bar{\upsilon})^2}$, in the frequency of the light observed; (c) Explain how measurements of the dispersion of a spectral line observed in starlight allow one to determine the temperature of the star.

2) (a) Stowe problem 18-11; (b) Stowe problem 18-12. Additional information needed to complete this problem: The mass of an air molecule is about 5×10^{-26} kg.

3) (a) Stowe problem 19-12; (b) Stowe problem 19-13.

4) Suppose that the molecules of a gas interact with each other through a radial force of magnitude $F = CR^{-s}$, where s is a positive integer.

(a) Use dimensional analysis to show how the total scattering cross section is related to the molecules' relative speed v, their masses m, and the force constant C; (b) How does the coefficient of viscosity of this gas depend on the temperature T?

5) A spacecraft in the shape of a cube of edge length L moves through space with a velocity $\vec{v}$ parallel to one of its edges. The spacecraft has mass M. The surrounding gas consists of molecules of mass m at temperature T; the number n of molecules per unit volume is small, so the mean free path of the molecules is much larger than L. Assume that collisions of the molecules with the spacecraft are elastic. Assume that $|\vec{v}|$ is small compared to the mean speed of the gas molecules. Estimate the mean retarding force exerted on the spacecraft by its collisions with the interplanetary gas. Ignore the distribution of velocities of the gas molecules. If the spacecraft is not subject to external forces other than the collisions with the gas, how long a time will elapse before the velocity of the craft is reduced to half its original value?

Answers to homework due 16 Nov. 2022

(1) $\nu = \nu_0\left(1+\frac{v_x}{c}\right)$

ν_0 and c are constants. Using Stowe Eq.

a) $\overline{\nu} = \nu_0\left(1+\frac{\overline{v_x}}{c}\right)$.

But $\overline{v_x} = 0$ (Stowe p. 314), so

$\boxed{\overline{\nu} = \nu_0}$

b) $\overline{\Delta\nu^2} = \overline{(\nu-\overline{\nu})^2} = \overline{(\nu-\nu_0)^2} = \overline{\left(\nu_0 + \frac{\nu_0 v_x}{c} - \nu_0\right)^2}$

$= \frac{\nu_0^2}{c^2}\overline{v_x^2}$

$\overline{v_x^2} = \int_{-\infty}^{+\infty} v_x^2 P(v_x)\, dv_x = \frac{kT}{m}$

So $\overline{\Delta\nu^2} = \boxed{\frac{\nu_0^2 kT}{mc^2}}$

c) The answer to part b) shows that measuring the rms broadening, $\Delta\nu = \sqrt{\overline{\Delta\nu^2}} = \frac{\nu_0}{c}\sqrt{\frac{kT}{m}}$ measures T.

The ν_0 can be found from the light's mean frequency ν_0 then given in.

2 a) $\overline{v_x} = \int_0^\infty P(v_x)\, v_x\, dv_x$

$$= \int_0^\infty \left(\frac{\beta m}{2\pi}\right)^{1/2} e^{-(\beta m/2)v_x^2}\, v_x\, dv_x$$

Use Stowe Eq 18A.4 with $\alpha = \left(\frac{\beta m}{2}\right)$ and $n = 0$

$$= \left(\frac{\beta m}{2\pi}\right)^{1/2} \frac{0!}{2\left(\frac{\beta m}{2}\right)} = \left(\frac{1}{2\pi\beta m}\right)^{1/2}$$

Plug in $\beta = \frac{1}{kT}$

$$= \boxed{\left(\frac{kT}{2\pi m}\right)^{1/2}}$$

b) $\rho = 2.7 \times 10^{25} / m^3$

$m = 5 \times 10^{-26}\ kg$

$T = 298\ K$

$$f_x = \rho\overline{v_x} = \rho\left(\frac{kT}{2\pi m}\right)^{1/2} =$$

$$\left(\frac{2.7\times10^{25}}{m^3}\right)\left[\frac{\left(1.381\times10^{-23}\frac{J}{K}\right)(298\ K)}{2\pi\left(5\times10^{-26}\ kg\right)}\right]^{1/2} = \boxed{3.1\times10^{27} / m^2 s}$$

c) Rate = flux × aperture

$$= \left(\frac{3.1\times10^{27}}{m^2\text{-}s}\right) \times \pi\,(1\times10^{-4})^2\,m^2 = \boxed{9.7\times10^{19}\ /\text{sec}}$$

(3) a) $R = 10^{-10}\,m$

$m = 5\times10^{-24}\,kg$

$\nu = 5$

$\rho = 2.7\times10^{25}\,/m^3$

$T = 300K$

$$D = (\text{const})\cdot\bar{v}\cdot l$$

$$\bar{v} = \left(\frac{8kT}{\pi m}\right)^{1/2}$$

$$l = \frac{1}{\sqrt{2}\,\rho\cdot 4\pi R^2}$$

$$\text{So } D = (\text{const})\left[\frac{8(1.381\times10^{-23}\,J/K)(300K)}{\pi(5\times10^{-26}\,kg)}\right]^{1/2}\frac{1}{4\pi\sqrt{2}\left(\frac{2.7\times10^{25}}{m^3}\right)(10^{-}}$$

$$\boxed{D = (\text{const})\cdot 9.6\times10^{-5}\,m^2/s}$$

$$K = (\text{const})\cdot\rho\cdot c_m\cdot\bar{v}\,l$$

$$c_m = \frac{\nu k}{2}$$

$$K = (\text{const})\cdot\left(\frac{2.7\times10^{25}}{m^3}\right)\cdot\frac{5}{2}\cdot(1.381\times10^{-23}\,J/K)\cdot\bar{v}\cdot l$$

$$\boxed{K = (\text{const})\cdot 9.0\times10^{-2}\,J/K\text{-}m\text{-}s}$$

$$\eta = (\text{const})\cdot\rho\cdot m\cdot\bar{v}\cdot l$$

$$\eta = (\text{const})\cdot\left(\frac{2.7\times10^{25}}{m^3}\right)(5\times10^{-26}\,kg)\cdot\bar{v}\cdot l$$

$$\boxed{\eta = (\text{const})\cdot 1.3\times10^{-4}\,kg/m\text{-}s}$$

Yes, the dimensions agree.

b) $Q_x = -K \frac{dT}{dx}$

$$= -(\text{const}) \cdot 9 \times 10^{-2} \frac{J}{K \cdot m \cdot s} \times \left(0.5 \frac{K}{m}\right) \cong \boxed{4.5 \times 10^{-2} \frac{J}{m^2 \cdot s}}$$

(4) a) The dimensions of a crossection σ are $(\text{length})^2$

If $F = CR^{-s}$, the dimensions of C are

$$C = FR^s = \frac{(\text{mass})(\text{length})}{(\text{time})^2} \cdot (\text{length})^s$$

$$= (\text{mass})(\text{length})^{s+1}(\text{time})^{-2}$$

The question asks what exponents "a", "b", "d" are required to make the following dimensionally correct:

$$\sigma = C^a \cdot m^b \, v^d$$

$$\downarrow$$

$$(\text{length})^2 = \left[\text{mass}\cdot\text{length}^{s+1}\cdot\text{time}^{-2}\right]^a \cdot [\text{mass}]^b \cdot [\text{length}\cdot\text{time}^{-1}]^d$$

this works if
$$\boxed{\begin{aligned} a &= 2/(s-1) \\ b &= 2/(1-s) \\ d &= 4/(1-s) \end{aligned}}$$

b) From Stowe Eq 19.24,

$\eta \propto \rho \cdot m \cdot \bar{v} \cdot l$, and $l \propto \frac{1}{\rho\sigma}$, and $\bar{v} \propto \sqrt{\frac{T}{m}}$,

so $\eta \propto \rho \cdot m \cdot \sqrt{\frac{T}{m}} \cdot \frac{1}{\rho \cdot \sigma} \Rightarrow \frac{1}{\sigma}\sqrt{Tm}$

Plug in $\sigma \propto (mv^2)^{-2/s-1}$ and $T \propto (mv^2)$, then

$$\eta \propto T^{\left(\frac{1}{2}+\frac{2}{s-1}\right)} = \boxed{T^{\frac{s+3}{2(s-1)}}}$$

(5) Because of the spacecraft's motion, more molecules strike its leading edge than its trailing edge. Each molecule transfers the same amount of momentum to the spacecraft.

a) Let v = Spacecraft velocity

V = molecules' mean speed

If there are n molecules per unit volume, then on average, $\frac{n}{6}$ are travelling in $+\hat{x}$, $\frac{n}{6}$ in $+\hat{y}$, $\frac{n}{6}$ in $+\hat{z}$,

$\frac{n}{6}$ in $-\hat{x}$, $\frac{n}{6}$ in $-\hat{y}$, and $\frac{n}{6}$ in $-\hat{z}$.

Suppose that the spacecraft is travelling in $+\hat{z}$. Then in time Δt its leading edge is struck by N_L molecules, where $N_L = \left(\frac{n}{6}\right) \cdot L^2 \cdot (V+v)\Delta t$.

The number of molecules that strike its trailing edge is

$$N_T = \left(\frac{n}{6}\right) L^2 \cdot (V-v)\Delta t$$

The momentum transferred from the molecules to the spacecraft in time Δt is

$$\Delta p = -2mV(N_L - N_T)$$

$$= -\frac{2}{3} m V n L^2 v \Delta t$$

$$\boxed{F = \frac{\Delta p}{\Delta t} = \frac{-2}{3} m V n L^2 v}$$

$$\downarrow$$

$$M \frac{dv}{dt} = \frac{-2}{3} m V n L^2 v$$

$$\downarrow$$

$$\frac{dv}{v} = \frac{-2}{3}\frac{m}{M} V n L^2 dt$$

$$\downarrow$$

$$v = v_0 \exp\left(\frac{-2}{3}\frac{m}{M} V n L^2 t\right)$$

When $\frac{v}{v_0} = \frac{1}{2}$,

$$\exp\left(\frac{-2}{3}\frac{m}{M} V n L^2 t\right) = \frac{1}{2},$$

$$\boxed{t = \frac{\ln 2}{\frac{2}{3}\frac{m}{M} V n L^2}}$$