Physics 301

Homework due 9 November 2022

1) Stowe problem 18-9.

2) A quantum mechanical harmonic oscillator has energy levels $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, where $n \in \{0,1,2,\ldots\}$, $\hbar$ is Planck's constant, and ω is the characteristic frequency for the oscillator and can be treated in this problem as a given constant. Suppose that the oscillator is in thermal contact with a reservoir at temperature T.

(a) Find the ratio between the probability that the oscillator is in its ground state and the probability that it is in its first excited state.

(b) Write an expression for the mean energy of the oscillator. The expression will involve a summation. You do not need to carry out the sum.

3) The Maxwell distribution was derived for a free particle. Consider the case in which the particle is not free but is instead confined by a gravitational potential such that its potential energy is $U = mgz$. Here m is the particle's mass, g is the gravitational constant, and z is the particle's height above earth. The particle still has kinetic energy as well.

(a) Derive a modified "Maxwell distribution" for this case.

(b) Find the probability $P(z)$ of finding the system at altitude z, as a function of $P(z=0)$, for an isothermal system.

4) What is the most probable kinetic energy $\tilde{E}$ of a molecule described by a Maxwellian velocity distribution? Is it equal to $\frac{m\tilde{v}^2}{2}$, where $\tilde{v}$ is the most probable speed of the molecule? Support your answer with a calculation.

5) Stowe problem 18-18.

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Answers to homework due 9 November 2022

1 a) $\overline{v^3} = \int_0^\infty v^3 P(v)\, dv$

$$= \int_0^\infty v^3 \cdot 4\pi \left(\frac{\beta m}{2\pi}\right)^{3/2} e^{-(\beta m/2)v^2}\, v^2\, dv$$

Stowe Stowe integral 18A.4, setting $x = v$

$\alpha = \frac{\beta m}{2}$

$n = 2$

Then $\overline{v^3} = 4\pi \left(\frac{\beta m}{2\pi}\right)^{3/2} \cdot \frac{2!}{2\left(\frac{\beta m}{2}\right)^3} = \boxed{4\pi \left(\frac{2kT}{m\pi}\right)^{3/2}}$

b) $\left(\overline{v^3}\right)^{1/3}$ for He:

Plug in $m = 6.6 \times 10^{-27}$ kg

$T = 298$ K

$k = 1.381 \times 10^{-23}$ J/K

$\left(\overline{v^3}\right)^{1/3} = \boxed{1464 \text{ m/s}}$

2 a) $E_n = (n+\frac{1}{2})\hbar\omega$

$$P_n = C e^{-\beta E_n}$$

$$\frac{P_1}{P_0} = \frac{e^{-\beta E_1}}{e^{-\beta E_0}} = e^{-\beta(E_1 - E_0)} = \exp\left[-\frac{1}{kT}\left(\frac{3}{2}-\frac{1}{2}\right)\hbar\omega\right]$$

$$= \boxed{\exp\left(-\frac{\hbar\omega}{kT}\right)}$$

b) $\overline{E} = \sum_{n=0}^{\infty} E_n P_n$

where $P_n = \dfrac{e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}} = \dfrac{e^{-(n+\frac{1}{2})\frac{\hbar\omega}{kT}}}{\sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\hbar\omega/kT}}$

3 a) Let $E = \frac{p^2}{2m} + mgz$

Then $P(\vec{r},\vec{p}) = Ce^{-\beta E} = Ce^{-\beta\left[\frac{p^2}{2m} + mgz\right]}$

$$= Ce^{-\beta\frac{p^2}{2m}} e^{-\beta mgz}$$

where $C = \int_{\vec{p}}\int_{\vec{r}} e^{-\frac{\beta p^2}{2m}} e^{-\beta mgz} d^3p\, d^3r$

To obtain a distribution that concerns $\vec{p}$ but not position

$$P(\vec{p}) = \int_{\vec{r}} Ce^{-\beta\left[p^2/2m - mgz\right]} d^3r$$

$$= \underbrace{C\int e^{+\beta mgz} d^3r}_{\text{call this } C'} \cdot e^{-\beta p^2/2m}$$

$$= \boxed{C' e^{-\beta p^2/2m}}$$

i.e. the modified distribution differs from the original only in normalization.

b.) $P(z) = \int_x\int_y\int_{\vec{p}} P(\vec{r},\vec{p})\, dx\, dy\, d^3p$

$$= \underbrace{C\int e^{-\beta p^2/2m} dx\, dy\, d^3p}_{\text{call this } C'} \cdot e^{-\beta mgz}$$

$$= \boxed{C' e^{-mgz/kT}}$$

By normalization, $C' = P(z=0)$

(4) $P(v)dv = 4\pi\left(\frac{\beta m}{2\pi}\right)^{3/2} e^{-\frac{\beta m v^2}{2}} v^2 dv$ (Stowe Eq. 18.8)

Let $E = \frac{mv^2}{2}$, so

$v^2 = \frac{2E}{m}$ and

$v = \left(\frac{2}{m}\right)^{1/2} E^{1/2}$

$dv = \frac{1}{2}\left(\frac{2}{m}\right)^{1/2} E^{-1/2} dE = \frac{dE}{\sqrt{2mE}}$

So $P(E)\,dE = 4\pi\left(\frac{\beta m}{2\pi}\right)^{3/2} e^{-\beta E} \cdot \frac{2E}{m} \frac{1}{\sqrt{2mE}} dE$

$$\frac{dP(E)}{dE} = 4\pi\left(\frac{\beta m}{2\pi}\right)^{3/2} \cdot \sqrt{\frac{2}{m^3}}\; \underbrace{\frac{1}{dE}\left[e^{-\beta E} \cdot E^{1/2}\right]}_{\downarrow} = 0 \text{ at max}$$

$$e^{-\beta E} \cdot \frac{1}{2} E^{-1/2} - \beta e^{-\beta E} E^{1/2} = 0$$

$$\frac{E^{-1/2}}{2} - \beta E^{1/2} = 0$$

$$\boxed{\tilde{E} = \frac{1}{2\beta} = \frac{kT}{2}}$$

If you do the same operation to Stowe Eq. 18.8, you find that $\tilde{v} = \sqrt{\frac{2kT}{m}}$, so $\frac{m\tilde{v}^2}{2} = kT \neq \tilde{E}$.

5 $T = 273.15\,K$

$p = 1$ atm

$\rho_{N_2} = 0.808 \frac{g}{cm^3}$

$\rho_{He} = 0.145 \frac{g}{cm^3}$

a) To find the radius, first find the volume:

$$\frac{\text{Volume}}{\text{molecule}} = \frac{1}{\rho} \times \frac{\text{Atomic weight}}{N_{Avogadro}}$$

For N_2:

$$\frac{\text{Volume}}{\text{molecule}} = \frac{cm^3}{0.808\,g} \times \frac{28\,g}{mole} \times \frac{mole}{6.02 \times 10^{23}\,\text{molecules}}$$

$$= 5.76 \times 10^{-23} \frac{cm^3}{\text{molecule}}$$

$\text{Volume} = \frac{4}{3}\pi R^3$, so $R = \left(\frac{3}{4\pi} \cdot \text{Volume}\right)^{1/3}$

$$R = 2.3 \times 10^{-8}\,cm = \boxed{2.3\,\text{Å}} \text{ for } N_2$$

A similar calculation for He yields $\boxed{R_{He} = 1.8\,\text{Å}}$

b) Mean free path: $l = \frac{1}{4\sqrt{2}\,\pi R^2 \rho}$

$\rho = N_{Avogadro} \times \frac{\text{mole}}{22.4 \text{ liters}} = \frac{6.02 \times 10^{23}}{22.4} = 2.69 \times 10^{22}/\text{liter}$

$= 2.69 \times 10^{25}/m^3$

So $l_{N_2} = \frac{1}{4\sqrt{2}\,\pi (2.3 \times 10^{-10})^2 m^2 \left(\frac{2.69 \times 10^{25}}{m^3}\right)} = \boxed{3.9 \times 10^{-8} m}$

A similar calculation for He yields $\boxed{l_{He} = 6.4 \times 10^{-8} m}$

c) $\nu_c = 4\sqrt{2}\,\pi R^2 \rho \bar{v}$

$\bar{v} = \left(\frac{8kT}{m\pi}\right)^{1/2}$

$m = \rho \cdot \frac{4}{3}\pi R^3$

For N_2, $m = \frac{0.808 g}{cm^3} \times \frac{4}{3}\pi (2.3 \times 10^{-8})^3 cm^3 = 4.1 \times 10^{-23} g$

So $\bar{v}_{N_2} = \left[\frac{8\left(1.381 \times 10^{-23} \frac{J}{K}\right)(273 K)}{(4.1 \times 10^{-26} kg)\,\pi}\right]^{1/2} = 484 \frac{m}{s}$

So $\nu_c^{(N_2)} = 4\sqrt{2}\,\pi (2.3 \times 10^{-10})^2 m^2 \left(\frac{2.69 \times 10^{25}}{m^3}\right)\left(484 \frac{m}{s}\right)$

$= \boxed{1.2 \times 10^{10}/sec}$

A similar calculation for He yields $\boxed{\nu_c^{(He)} = 2.6 \times 10^{10}/sec}$