Physics 301

Homework due 2 November 2022

1) Stowe problem 16-5.

2) Stowe problem 16-17.

3) Suppose that the energy stored at any instant in a certain degree of freedom is linear, rather than quadratic, in the generalized coordinate; that is, suppose $E = cq$, where E is the instantaneous energy, q is the position or momentum, and c is a constant. Show that the average energy for this degree of freedom is $E = kT$.

4) Stowe problem 17-6.

5) Stowe problem 17-8.

Physics 301

Answers to homework due 2 Nov 2022

1.) $T = 500K$

$\mu = 0$

$\varepsilon_s = 0.2eV$

a) $\beta(\varepsilon_s - \mu) = \frac{(\varepsilon_s - \mu)}{kT} = \frac{0.2eV - 0\,eV}{(8.63 \times 10^{-5}\frac{eV}{K})(500K)} = \boxed{4.63}$

b) $C = \left[\sum_n e^{-n\beta(\varepsilon_s - \mu)}\right]^{-1}$

$= \left[\sum_{n=0}^{\infty} e^{-4.63n}\right]^{-1}$

$= \left[\frac{1}{1 - e^{-4.63}}\right]^{-1} = \boxed{0.990}$

c) $P(n=3) = \frac{e^{-3\beta(\varepsilon_s - \mu)}}{\sum_{n=0}^{\infty} e^{-n\beta(\varepsilon_s - \mu)}} = (0.990)\,e^{-3(4.63)} = \boxed{9.18 \times 10^{-7}}$

(2) Recall from E&M that the relative energies of the 3 leve[ls] are given by $E = -\vec{\mu}\cdot\vec{B}$. So for $\vec{B} = B_z\hat{z}$ only, $E = -\mu_z$

For $B_z = 1\,T$, and using $\mu_0 = 0.93\times10^{-23}\,J/T$, the energies

Level # (s)	μ_z value	$E_s = -\mu_z B_z$
0	$1\mu_0$	$-(1\mu_0)\cdot(1T) = -0.93\times10^{-23}\,J$
1	0	$0\,J$
2	$-1\mu_0$	$-(-1\mu_0)(1T) = +0.93\times10^{-23}\,J$

a) $$P_s = \frac{e^{-\beta E_s}}{\sum_{s=0}^{2} e^{-\beta E_s}}$$

To calculate the denominator:

$$\beta = \frac{1}{kT} = \frac{1}{\left(1.381\times10^{-23}\frac{J}{K}\right)(300K)} = 2.41\times10^{20}\,J^{-1}$$

$$\sum_{s=0}^{2} e^{-\beta E_s} = \exp\left[-(2.41\times10^{20})\times(-0.93\times10^{-23})\right] +$$

$$\exp\left[-(2.41\times10^{20})\times(0)\right] +$$

$$\exp\left[-(2.41\times10^{20})\times(+0.93\times10^{-23})\right]$$

$$= 1.0022 + 1 + 0.9978 = 3.0000$$

So $P_{s=0} = \frac{e^{-\beta E_0}}{3} = \frac{1}{3}\exp\left[-(2.41\times10^{20})(-0.93\times10^{-23})\right] =$

$= \frac{1.0022}{3} = \boxed{0.3341}$

$P_{s=1} = \frac{e^{-\beta E_1}}{3} = \frac{1}{3}\exp\left[(-2.41\times10^{20})(0)\right] = \frac{1}{3}e^0 = \boxed{\frac{1}{3}}$

$P_{s=2} = \frac{e^{-\beta E_2}}{3} = \frac{1}{3}\exp\left[-(2.41\times10^{20})(+0.93\times10^{-23})\right] =$

$= \frac{0.9978}{3} = \boxed{0.3326}$

b) $\overline{\mu_z} = \sum_s P_s \mu_{z_s} = (0.3341)(1\mu_0) + (0.333)(0) + (0.3326)(-1\mu_0)$

$= 0.0015\,\mu_0 = \boxed{1.4\times10^{-26}\ J/T}$

c) $\mu_{TOT} = N\overline{\mu_z} = (6.02\times10^{23})\left(1.4\times10^{-26}\frac{J}{T}\right) = \boxed{8.4\times10^{-3}\frac{J}{T}}$

(3) The following derivation uses the procedure shown in the lecture. The student may also use the procedure in Stowe section 17.A.

Begin with the definition of the mean:

$$\overline{E} = \sum_s P_s E_s \qquad \text{where } s \text{: indexes states}$$

Plug in $P_s = \dfrac{e^{-\beta E_s}}{\sum_s e^{-\beta E_s}}$; $\beta = \dfrac{1}{kT}$

Replace $\sum$ by $\int dq$

Replace E_s by cq

$$\overline{E} = \frac{\int dq\, e^{-\beta cq} \cdot cq}{\int dq\, e^{-\beta cq}}$$

$$= -\frac{d}{d\beta} \ln\left[\int dq\, e^{-\beta cq}\right]$$

Let $y = q\beta$, so

$q = \dfrac{y}{\beta}$ and

$\beta cq = cy$ and

$dq = \frac{1}{\beta} dy$

Then $\overline{E} = -\frac{d}{d\beta} \ln\left[\int \frac{1}{\beta} dy\, e^{-cy}\right]$

$$= -\frac{d}{d\beta}\left\{\ln\frac{1}{\beta} + \underbrace{\ln\int dy\, e^{-cy}}\right]$$

once this integral is eval
it is independent of β

$$= -\frac{d}{d\beta}\left(\ln\frac{1}{\beta}\right)$$

$$= -\frac{d}{d\beta}\left(-\ln\beta\right)$$

$$= \frac{d}{d\beta}\left(\ln\beta\right)$$

$$= \frac{1}{\beta}$$

Plug in $\beta = \frac{1}{kT}$

So $\boxed{\overline{E} = \frac{1}{\frac{1}{kT}} = kT}$

(4) $m = 0.5\,kg$
$l = 2m$
$T = 25°C = 298K$

The pendulum has 2 degrees of freedom (due to PE + KE in its direction of motion)

a) $\overline{E}_{thermal} = (\#\,d.o.f.) \times \frac{1}{2}kT$

$$= kT = \left(1.381\times10^{-23}\frac{J}{K}\right)(298K) = \boxed{4.1\times10^{-21}\,J}$$

b) Its horizontal displacement x is associated with the degree of freedom having energy $E_{potential} = \frac{1}{2}k_s x^2$.
(k_s is the spring constant; $k_s = mg/l$)
Working in analogy to the text on Stowe page 307:

$\overline{E_{potential}} = \frac{1}{2}k_s\overline{x^2} = \frac{1}{2}k_B T$, so

$$\left(\overline{x^2}\right)^{1/2} = \left[\frac{k_B T}{k_s}\right]^{1/2}$$

$$= \left[\frac{\left(1.381\times10^{-23}\frac{J}{K}\right)(298K)}{\frac{(0.5kg)\left(9.8\frac{m}{sec^2}\right)}{(2m)}}\right]^{1/2} = \boxed{4.1\times10^{-11}\,m}$$

5) a) $E_{vib} = 0.2\,eV$

Ignore vibrational mode if $kT \ll (E_{vib} - E_{ground})$

(let $E_{ground} = 0$)

$$T \ll \frac{E_{vib}}{k}$$

$$T \ll \frac{0.2\,eV}{8.63\times10^{-5}\,\frac{eV}{K}} = \boxed{2317\,K}$$

b) $I = 10^{-45}\,kg\text{-}m^2$

Ignore rotation if $kT \ll (E_{rot} - E_{ground})$

$E_{rot} = \frac{\hbar^2}{2I}$

let $E_{ground} = 0$

$$T \ll \frac{\hbar^2}{2Ik}$$

$$T \ll \frac{(1.06\times10^{-34})^2\,J^2\text{-}s^2}{2(10^{-45}\,kg\text{-}m^2)\left(1.381\times10^{-23}\,\frac{J}{K}\right)}$$

$$\boxed{T \ll 0.406\,K}$$

c) Room temperature $\approx 298\,K$.

At this temperature, vibrational modes are inactive but rotational modes are not. So there are 6N degrees of freedom per molecule (N = # atoms per molecule)

Following Stowe p 303:

$$E = N \cdot \frac{\nu}{2} kT = \frac{6}{2} kT = 3nRT$$

$$\Delta E = \Delta Q - p\Delta V$$

$$C_v = \left.\frac{dQ}{dT}\right)_V = \left.\frac{dE}{dT}\right)_V = 3nR$$

So $c_v = \frac{C_v}{n} = 3R = 3 \cdot \left(8.31 \frac{J}{K\text{-mole}}\right) = \boxed{24.93 \frac{J}{K\text{-mole}}}$