III. Fermions and bosons

It is a fact of nature that particles come in two varieties:

fermions - only 1 is permitted per state

bosons - the number per state is unrestricted, but integer

These requirements correspond to different classes of solutions to the Dirac Equation (the relativistic version of the Schroedinger Equation, including requirements on Lorentz covariance and causality).

We now find $\bar{n}$ for each of these two varieties.

Finding $\bar{n}$ for fermions: the sum can run only over the range 0 to 1.

$$\sum_{n=0}^{1} e^{-nx} = 1 + e^{-x}$$

Thus, $\bar{n}_f = -\frac{\partial}{\partial x}\left[\ln\left(\sum_{n=0}^{1} e^{-nx}\right)\right] = -\frac{\partial}{\partial x}\ln\left(1+e^{-x}\right)$

$$= -\frac{1}{1+e^{-x}}\cdot\left(-e^{-x}\right) = \frac{e^{-x}}{1+e^{-x}}$$

To put this into an alternative form, multiply by $\frac{e^{x}}{e^{x}}$:

$$\bar{n}_f = \frac{1}{e^{x}+1}$$

Remember $x = \beta\left(\varepsilon_s - \mu\right) = \frac{\varepsilon_s - \mu}{kT}$, so,

$$\bar{n}_f = \frac{1}{e^{(\varepsilon_s-\mu)/kT}+1}$$

Finding $\bar{n}$ for bosons: the sum must run over the range 0 to infinity.

$$\sum_{n=0}^{\infty} e^{-nx} = \frac{1}{1-e^{-x}}$$

(See Stowe Appendix 21A on how to compute this sum.)

$$\text{So } \bar{n}_b = -\frac{\partial}{\partial x}\left[\ln\left(\sum_{n=0}^{\infty} e^{-nx}\right)\right] = -\frac{\partial}{\partial x}\ln\left(\frac{1}{1-e^{-x}}\right) = -\frac{\partial}{\partial x}\left[\ln 1 - \ln\left(1-e^{-x}\right)\right]$$

$$= \frac{\partial}{\partial x}\ln\left(1-e^{-x}\right) = \frac{1}{1-e^{-x}}\cdot\left(e^{-x}\right) = \frac{e^{-x}}{1-e^{-x}}$$

To put this into an alternative form, multiply by $\frac{e^{x}}{e^{x}}$:

$$\bar{n}_b = \frac{1}{e^{x}-1} = \frac{1}{e^{(\varepsilon_s-\mu)/kT}-1}$$

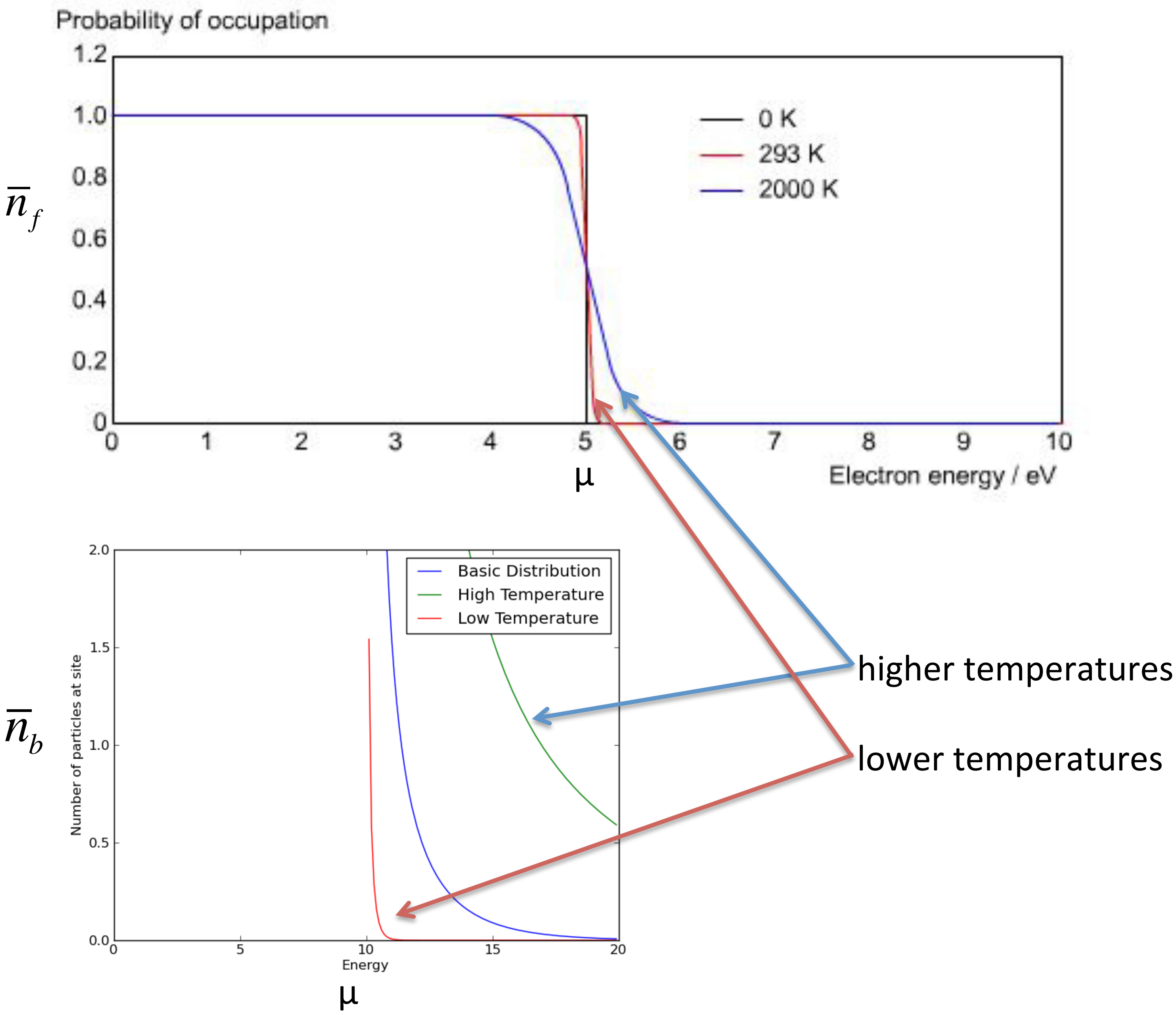
A minus sign, where $\bar{n}_f$ has a plus sign.

This seemingly small difference will have amazing consequences.

Notice that $\bar{n}$ does not tell us what states exist, nor which are accessible.

It only gives the average number of particles in a state, if that state exists.

What the $\bar{n}$ distributions look like:



higher temperatures

lower temperatures

I. The relationship between classical and quantum statistics

II. Bosons aggregate, fermions disperse

III. In what cases do classical statistics apply?

Please read Stowe chapter 26.

I. The relationship between classical and quantum statistics

Recall occupation numbers, the mean number of particles in a state:

$$\bar{n}_f = \frac{1}{e^{(\varepsilon_s-\mu)/kT}+1}$$

fermions: particles that constitute matter (e.g., electrons, quarks)

$$\bar{n}_b = \frac{1}{e^{(\varepsilon_s-\mu)/kT}-1}$$

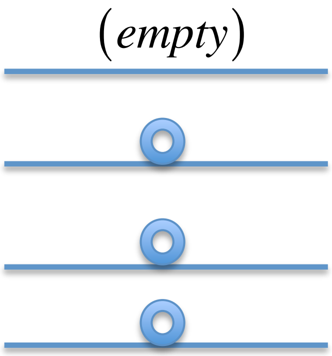
bosons: particles that transmit forces (e.g., photons)

Quantum mechanical effects become evident differently for fermions and bosons.

(1) How QM effects make fermion statistics different from classical statistics:

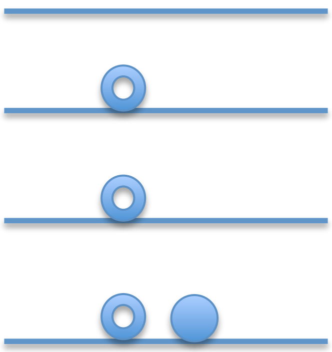
When >1 fermion is present, the Pauli Exclusion Principle forces fermions into higher states than would be required for classical particles.

Consider 4 energy levels, 3 of which have 1 particle in each:

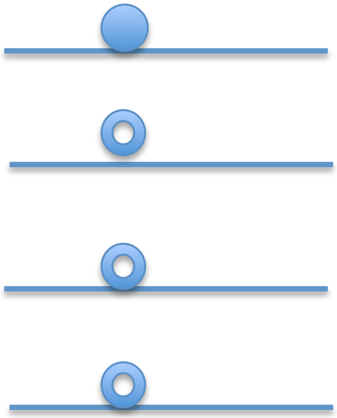


Add a new, lower energy, particle " ● " to the system.

If they are classical (distinguishable) particles, it can settle into the lowest level:



If they are indistinguishable fermions, the Pauli principle forces it into an unoccupied level:



So we expect that $\bar{n}_f < \bar{n}_{classical}$ and $\bar{E}_f > \bar{E}_{classical}$.

But if the particle we wish to add to the system has an energy that is much higher than the environment ($E >> kT$), it will enter a high level regardless of whether it obeys classical or Fermi statistics. Thus,
for $E >> kT$, we expect that: $\bar{n}_f \approx \bar{n}_{classical}$

To see this mathematically, note:

$$\bar{n}_f = \frac{1}{e^{(\varepsilon_s-\mu)/kT}+1}$$

When $\varepsilon_s >> kT$, $e^{(\varepsilon_s-\mu)/kT} >> 1$, so we can neglect the "+1" in the denominator.

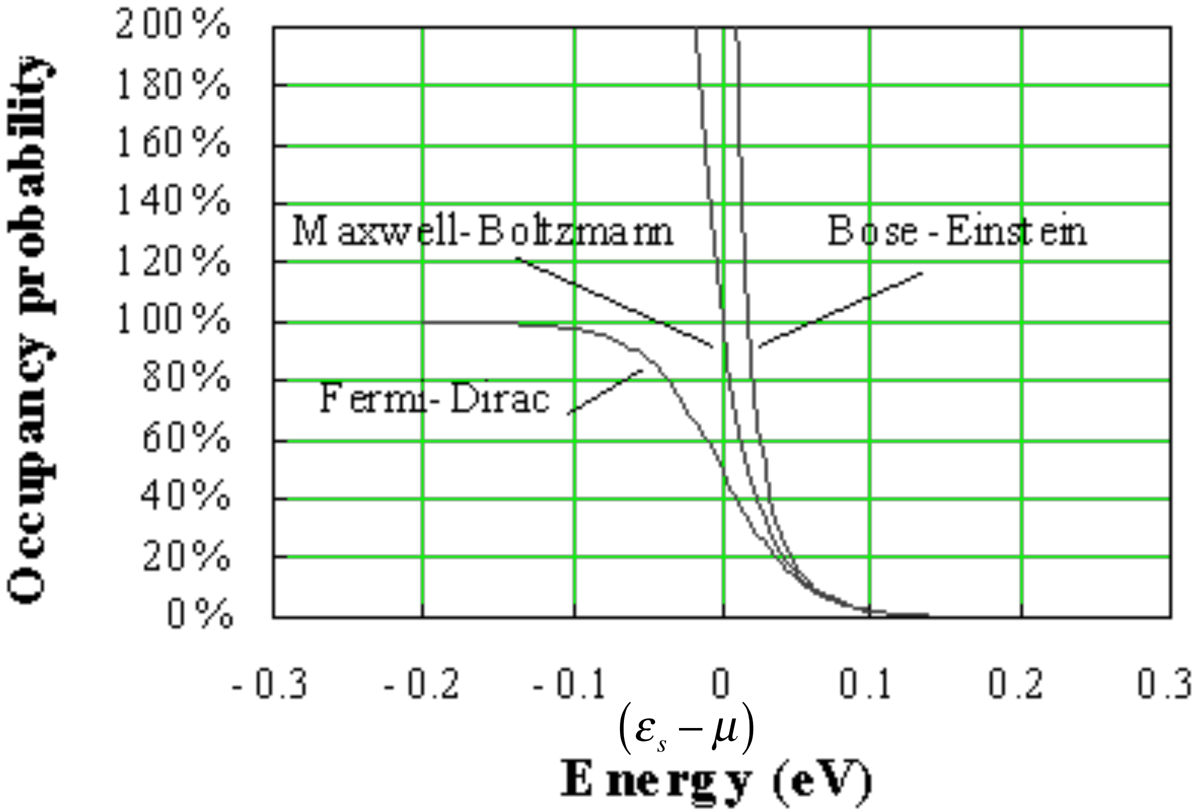
Then,

$$\bar{n}_f \Rightarrow \frac{1}{e^{(\varepsilon_s-\mu)/kT}} = \text{"}\bar{n}_{MB}\text{"}$$, the Maxwell-Boltzmann distribution

We can also call this "$\bar{n}_{classical}$", the statistical distribution for classical particles.

Compare the Fermi and classical ("M-B," (Maxwell-Boltzmann) distributions:

As intuited, $\bar{n}_{classical} > \bar{n}_f$ until $E >> kT$.



When $\varepsilon_s << \mu$, $e^{(\varepsilon_s - \mu)/kT}$ is very small,

so $\dfrac{1}{e^{(\varepsilon_s-\mu)/kT}+1} \approx \dfrac{1}{0+1} = 1$ for fermions

But:

$\dfrac{1}{e^{(\varepsilon_s-\mu)/kT}} \to \dfrac{1}{0} \to \infty$ for classical (M-B) particles.

When $\varepsilon_s >> \mu$, $e^{(\varepsilon_s - \mu)/kT} >> 1$

so $\dfrac{1}{e^{(\varepsilon_s-\mu)/kT}+1} \approx \dfrac{1}{e^{(\varepsilon_s-\mu)/kT}}$ for fermions

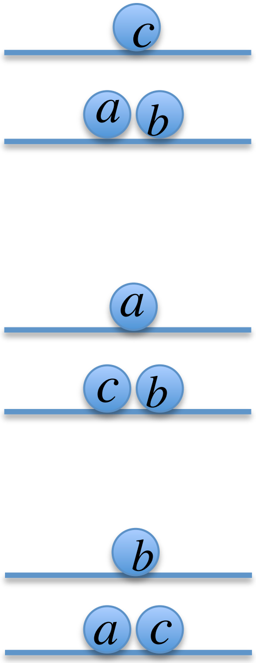
So in this case, the fermion and M-B distributions converge.

(2) How QM effects make boson statistics different from classical statistics:

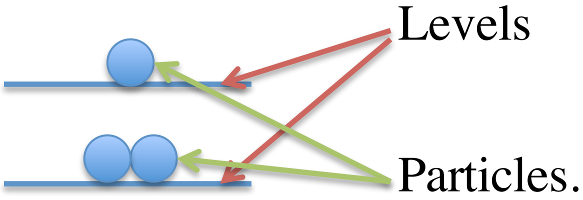
Recall bosons are like classical particles in that more than one can be present in a state. They are different because once the particles have selected their states, permutations among (identical) bosons don't result in new states, but permutations among (distinguishable, non-identical) classical particles do.

Example:

Classical



Bosons



Levels

Particles.

Thus: 3 states if classical, 1 state if bosons.

So if we were using the Maxwell-Boltzmann (classical) distribution as an approximation to the boson distribution, we should divide the resulting # states by 3.

Because the partition function depends upon number of states,

(recall $Z \equiv \sum_{\#\, states} e^{-\beta E_s}$) we should correct Z by a factor of 3, also.

Recall (lecture slide 286) we found that

$$Z_{\substack{indistinguishable \\ particles}} = \frac{1}{N!} Z^N_{\substack{distinguishable \\ particles}}$$

Note "distinguishable particles" are the classical, Maxwell-Boltzmann case.

Consider the case where the "indistinguishable particles" are bosons with > 1 boson per level. In this example, the $\frac{1}{N!}$ correction will be too severe.

We see that the true correction would be $\frac{1}{3}$, but $\frac{1}{N!} = \frac{1}{3!} = \frac{1}{6}$.

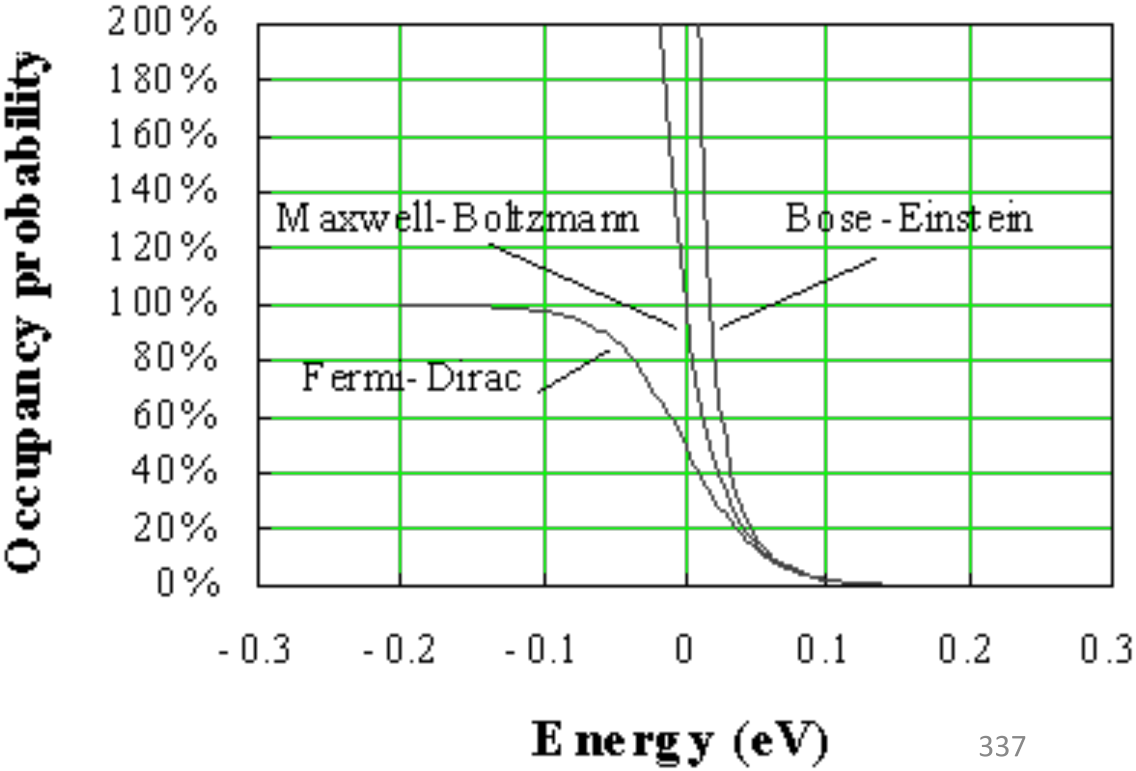
We see that: when there is high probability of multiple occupancy of states (these are the low energy states), then the $\frac{1}{N!}$ classical correction is too severe for bosons.

The $\frac{1}{N!}$ classical factor formally applies only if there is no multiple occupancy of states. This is the case for very high energies.

Thus:

At high energies, the classical and boson distributions converge.

At low energies, where multiple particles are more likely to occupy the same level, $\bar{n}_{\substack{Maxwell-Boltzmann \\ (classical)}} < \bar{n}_{boson}$ due to the over-correction by the $\frac{1}{N!}$ factor.



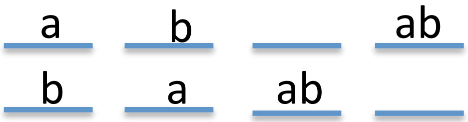
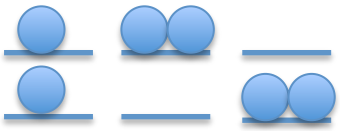
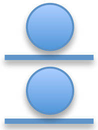
II. Bosons aggregate, fermions disperse

This is a direct consequence of the statistical distributions.

$$\text{Recall Prob}\begin{pmatrix}\text{observing a particular property} \\ \text{or configuration}\end{pmatrix} = \frac{\text{\# states with that property}}{\text{total \# states}}$$

(Recall the example table on lecture slide 76.)

Consider 2 levels and 2 particles:

	Classical (distinguishable)	Bosons	Fermions
Possible arrangements			
Ω_{tot}	4	3	1
# double occupancy states Ω_0	2	2	0
Prob(double occupancy)	$\frac{2}{4} = 0.5$	$\frac{2}{3} = 0.67$	$\frac{0}{1} = 0$

Define "clustering": increased probability of double occupancy.

Then we see that $P_{cluster}(bosons) > P_{cluster}(classical) > P_{cluster}(fermions)$

III. In what cases do classical statistics apply?

Classical stats approximate to both the Bose-Einstein and the Fermi-Dirac stats, when

$$\frac{\text{\# states available to a particle}}{\text{\# particles}} >> 1.$$

The # states may be available because

i) the particle's energy is high, or

ii) the particle density is low.

Consider a particle in volumes $V_r = R^3$ and

$$V_p = \langle p_{rms} \rangle^3 \equiv \langle p \rangle^3$$

$$\frac{\text{\# states available}}{\text{\# particles}} = \text{\# states per particle} \approx \frac{V_r V_p}{h^3} = \frac{R^3 \langle p \rangle^3}{h^3}$$

So classical stats is applicable when $\frac{R^3 \langle p \rangle^3}{h^3} >> 1$

$$R\langle p \rangle >> h \qquad \text{"Eq 1"}$$

We now work out what conditions this implies for volume V, number N, temperature T.

Recall the Equipartition Theorem: for free, non-relativistic particles,

$$\overline{KE} = 3\times\frac{1}{2}kT \qquad (3 = \#\,\text{dof, from } p_x, p_y, p_z)$$

$$\frac{1}{2m}\overline{p^2} = \frac{3}{2}kT$$

$$\overline{p^2} = 3mkT$$

$$p_{rms} = \langle p\rangle = \left(\overline{p^2}\right)^{1/2} = \left(3mkT\right)^{1/2}$$

Also, R^3 = volume per particle $= \dfrac{V}{N}$,

so $R = \left(\dfrac{V}{N}\right)^{1/3}$

Plug these into Eq 1:

Classical stats are applicable when $\left(\dfrac{V}{N}\right)^{1/3}\cdot\left(3mkT\right)^{1/2} >> h.$

I. Introduction to blackbody radiation

II. Quantum particle distributions

III. Photon distribution

IV. The Planck spectral distribution for a black body

I. Introduction to blackbody radiation

We are speaking here about a "quantum gas."

"Quantum" because the average particle's energy is so low that Fermi-Dirac or Bose-Einstein statistics must be used.

"Gas" to describe any group of particles whose states are so closely spaced that they are approximately continuous. The close spacing comes from the usual gas properties of low density and weak interaction.

Examples of quantum gases:

i) electrons in a conductor (Fermi-Dirac)

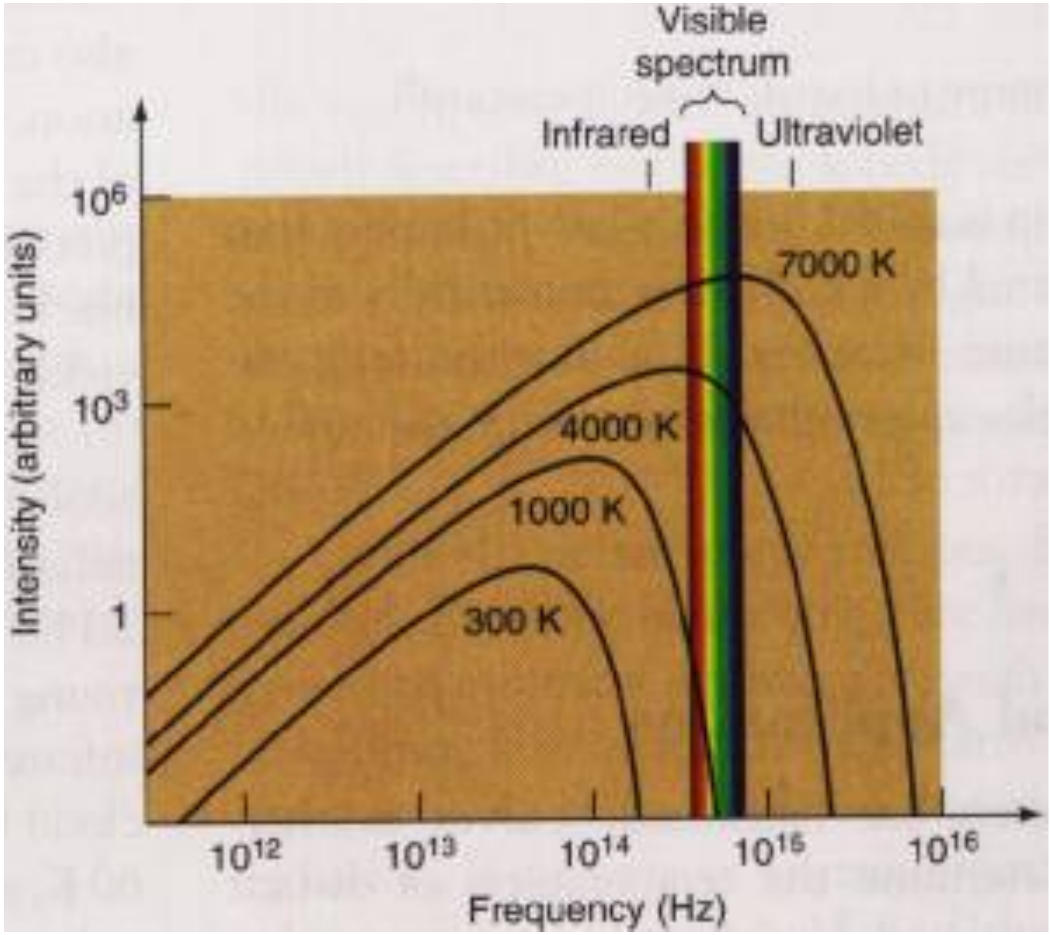
ii) liquid He atoms (Bose-Einstein)

iii) photons in an oven (Bose-Einstein)

iv) phonons: vibrations in a solid lattice (Bose-Einstein)

Why this is interesting:

Consider (iii), the photons in an oven. The observation of the oven's radiation frequency spectrum led Planck to hypothesize the quantization of energy (i.e. the existence of photons). That frequency spectrum looks like this:



Our goal: understand energy density $\frac{dE}{dVol} \equiv du = \left(\frac{\hbar}{\pi^2 c^3}\right)\frac{\omega^3 d\omega}{e^{\beta\hbar\omega}-1}$, where $\omega =$ frequency.

Steps needed:

i) Find number dN of particles per available phase space.

$$dN = \left(\text{\# quantum states}\right) \times \left(\text{mean \# particles per state}\right)$$

ii) Consider photons: spin 1, relativistic

iii) Find dE = energy per quantum state.

$$dE = dN \times \varepsilon = \left(\text{\#photons}\right) \times \left(\text{energy per photon}\right)$$

iv) Compute du = energy density $= \dfrac{dE}{dVol}$

II. Quantum particle distributions

Goal #1: find dN = # particles in the quantum state specified by energy ε in range dε.

$$dN = \underbrace{\left[\text{\#states in range d}\varepsilon\right]}_{\substack{\left(\text{density of states}\right)\times\left(d\varepsilon\right) \\ g\left(\varepsilon\right)\times d\varepsilon}} \times \underbrace{\left[\text{typical \# of particles per state}\right]}_{\substack{\text{occupation number} \\ n\left(\varepsilon\right)}}$$

Recall we showed (lecture slide 45) that $g\left(\varepsilon\right)d\varepsilon = \dfrac{d^3r d^3p}{h^3}$.

This is the number of states of a spinless particle.

Spin is a separate degree of freedom that exists in a space other than physical coordinate space.

Recall that if a system has n dof's which each make available Ω_i states, then the total #states $\Omega_0 = \prod_{i=1}^{n} \Omega_i$.

This Ω_0 is $g\left(\varepsilon\right)d\varepsilon$.

For simplicity of expression, combine all of the (r, p) dof's into one group, and all spin dof's into another group, so:

$$g(\varepsilon)d\varepsilon = \Omega_0 = \Omega_{r,p} \cdot \Omega_{spin}$$

$$= \frac{d^3 r d^3 p}{h^3} \cdot \Omega_{spin} \qquad \text{(Stowe calls } \Omega_{spin} \text{ "}\lambda\text{")}$$

What formula to use for Ω_{spin} :

Massive particles with spin j have $\Omega_{spin} = 2j + 1$

Massless particles with spin j have $\Omega_{spin} = 2j$

Examples:

Particle	Massive?	Spin	Ω_{spin}
photon	no	1	2
electron	yes	$\frac{1}{2}$	2

Rewrite, letting $\Omega_{spin} \equiv \lambda$.

If the particles are in a field-free region, $g(\varepsilon)d\varepsilon$ does not depend on position, so integrate over $\mathrm{d}^3 r$ to obtain volume V.

Then:

$$g(\varepsilon)d\varepsilon = \frac{\lambda V d^3 p}{h^3}.$$

$$d^3 p = p^2 dp \sin\theta d\theta d\phi.$$

Again for a field-free region, we can integrate over (θ,ϕ) to get 4π.

$$g(\varepsilon)d\varepsilon = \frac{\lambda V 4\pi p^2 dp}{h^3}.$$

Recall $dN = g(\varepsilon)d\varepsilon \cdot n(\varepsilon)$, so

$$dN = \frac{\lambda V 4\pi p^2 dp}{h^3} \cdot n(\varepsilon)$$

This is the number of particles in the phase space interval dp, centered on energy ε, for the arbitrary number of spin degrees of freedom = λ and physical volume V.

III. Photon distribution

Begin with the number of particles:

$$dN = \frac{\lambda V 4\pi p^2 dp}{h^3} \cdot n(\varepsilon) \qquad \text{"Eq 1"}$$

Consider photons:

1) they are bosons, so $n(\varepsilon) = \dfrac{1}{e^{\beta(\varepsilon-\mu)} - 1}$

2) they don't interact with each other, so $\mu = 0$.

3) Their $\lambda = \Omega_{spin} = 2$

4) they are relativistic, so their $\varepsilon = pc$.

Thus $p = \dfrac{\varepsilon}{c}$, so $p^2 = \dfrac{1}{c^2}\varepsilon^2$ and $dp = \dfrac{1}{c}d\varepsilon$.

Plug these into Eq 1 to get:

$$dN_{photons} = \frac{2 \cdot V \cdot 4\pi}{h^3} \cdot \frac{\varepsilon^2}{c^2} \cdot \frac{d\varepsilon}{c} \cdot \frac{1}{e^{\beta(\varepsilon-0)} - 1} = \frac{8\pi V}{h^3 c^3}\varepsilon^2 d\varepsilon \cdot \frac{1}{e^{\beta\varepsilon} - 1}$$

IV. The Planck spectral distribution for a black body

$$dN_{photons} = \frac{8\pi V}{h^3c^3}\varepsilon^2 d\varepsilon \cdot \frac{1}{e^{\beta\varepsilon}-1}$$

What this means: The # photons present is related to the energy per photon.

If photons are in equilibrium with their environment, their average energy is the same as the energy of the environment, which is proportional to the temperature.

Thus: the # photons present is related to the temperature of their environment.

Now find the energy E of the photon "gas".

Let $\varepsilon =$ energy per photon

$$dE = dN \cdot \varepsilon$$

$$= \frac{8\pi V}{h^3c^3}\varepsilon^2 d\varepsilon \cdot \frac{1}{e^{\beta\varepsilon}-1} \cdot \varepsilon$$

$$= \frac{8\pi V}{h^3c^3}\varepsilon^3 d\varepsilon \cdot \frac{1}{e^{\beta\varepsilon}-1}$$

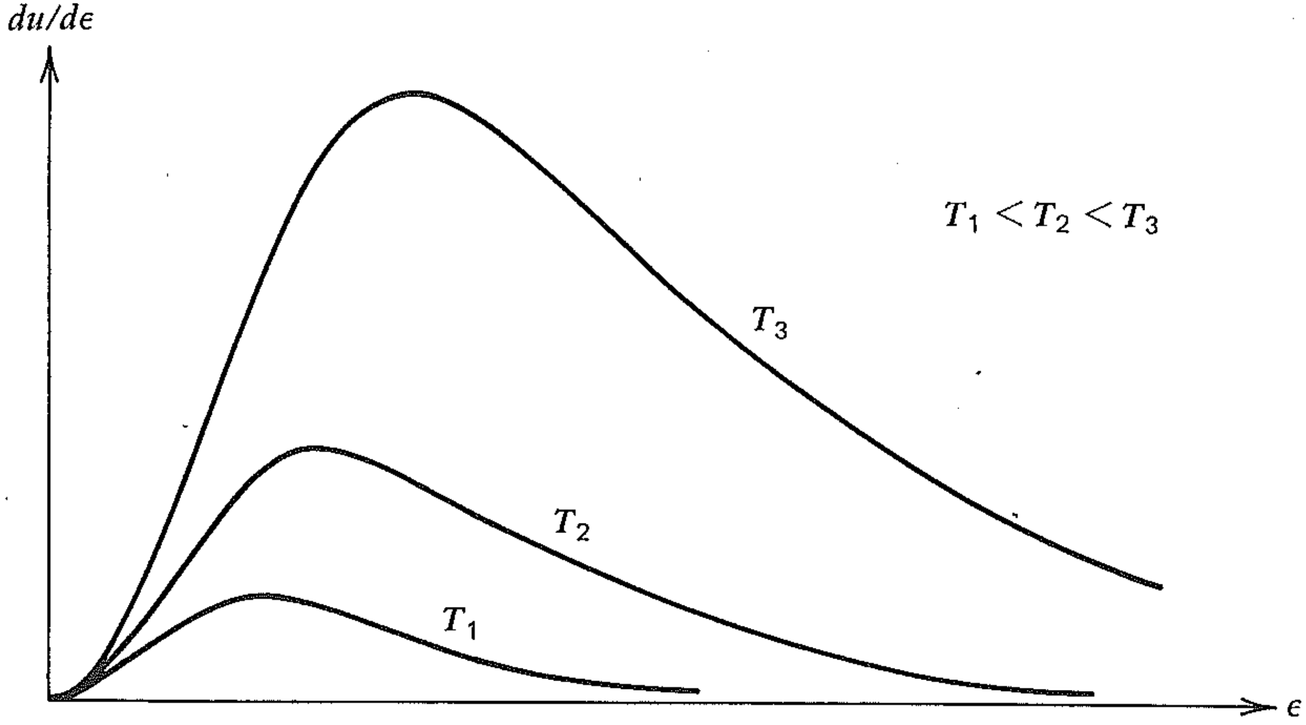
Take out the volume dependence by computing energy per volume:

photon energy density $du = \frac{dE}{dV} = \frac{8\pi}{h^3c^3}\varepsilon^3 d\varepsilon \cdot \frac{1}{e^{\beta\varepsilon}-1}$

Now find the energy density per range dε, centered on ε:

$$\frac{du}{d\varepsilon} = \frac{8\pi}{h^3c^3}\varepsilon^3 \cdot \frac{1}{e^{\beta\varepsilon}-1} \qquad \text{"Eq 1"}$$

Remember $\beta \equiv \dfrac{1}{kT}$ so this distribution is temperature-dependent:



It is a fact about nature that photon energy ε, which is the same as electromagnetic wave energy, is related to the wave frequency ω through

$$\varepsilon = \hbar\omega.$$

This is surprising because matter waves (i.e., water, sound, and so forth) do NOT have this relationship. For them, $\varepsilon \propto$ amplitude, not frequency.

Since $\varepsilon = \hbar\omega = \frac{h}{2\pi}\omega$,

$$d\varepsilon = \frac{h}{2\pi}d\omega$$

Plug these into du:

$$du = \frac{8\pi}{h^3c^3}\varepsilon^3 d\varepsilon \cdot \frac{1}{e^{\beta\varepsilon}-1} = \frac{8\pi}{h^3c^3}\cdot\frac{h^3\omega^3}{8\pi^3}\cdot\frac{hd\omega}{2\pi}\cdot\frac{1}{e^{\beta\hbar\omega}-1}$$

Then

$$\frac{du}{d\omega} = \frac{\hbar\omega^3}{\pi^2c^3}\cdot\frac{1}{e^{\beta\hbar\omega}-1}$$

This is the Planck law for the spectral distribution of photons in a thermal cavity.

To obtain an alternative form of the graph:

Note that since $\beta = \frac{1}{kT}$, $\beta kT = 1$. Cube this to get

$\beta^3 k^3 T^3 = 1$.

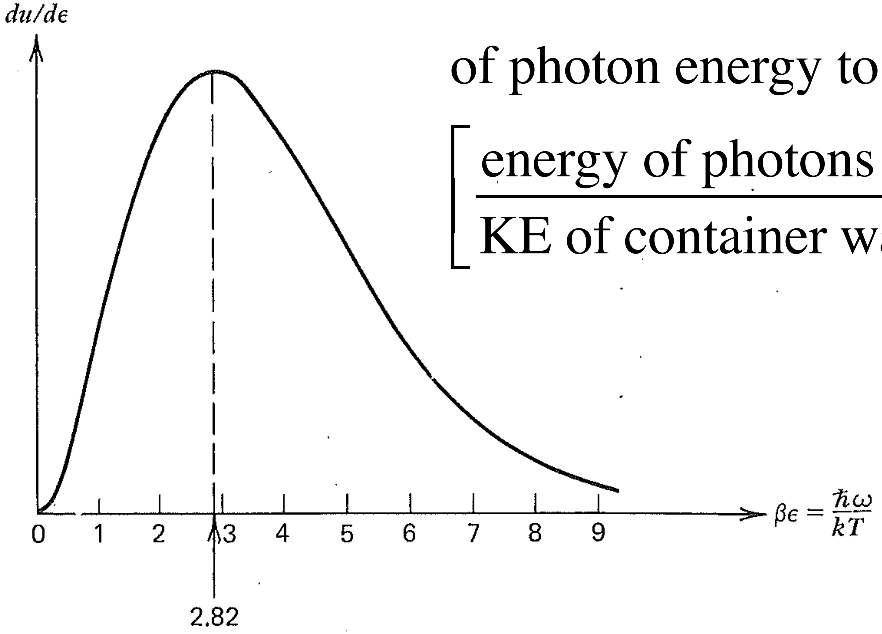
Multiply Eq 1 by "1" on the LHS and $\beta^3 k^3 T^3$ on the RHS.

$$1 \times \frac{du}{d\varepsilon} = \frac{8\pi}{h^3 c^3}\varepsilon^3 \cdot \frac{1}{e^{\beta\varepsilon} - 1} \times \beta^3 k^3 T^3$$

$$\frac{du}{d\varepsilon} = \left(\frac{8\pi k^3}{h^3 c^3}\right) \cdot \frac{T^3 (\beta\varepsilon)^3}{e^{\beta\varepsilon} - 1}$$

The horizontal axis $\beta\varepsilon = \frac{\hbar\omega}{kT}$ is the dimensionless ratio of photon energy to thermal energy, i.e. the ratio:

$$\left[\frac{\text{energy of photons emitted by hot container walls}}{\text{KE of container wall atoms that emit the photons}}\right]$$



Call $\omega\left(\frac{du}{d\varepsilon} \text{ is max}\right) \equiv \tilde{\omega}$

Call $T\left(\frac{du}{d\varepsilon} \text{ is max}\right) \equiv \tilde{T}$

Then $\frac{\tilde{\omega}}{\tilde{T}} = c\text{onstant} = 2.82$ "Wien's displacement law"

This means: if we increase T, $\tilde{T}$ increases and $\tilde{\omega}$ follows linearly.

I. The energy density of a photon gas

II. The Stefan-Boltzmann Law

III. Reflection, absorption, and emission by thermal bodies

I. The energy density of a photon gas

Recall $\dfrac{du}{d\varepsilon} = \dfrac{8\pi}{h^3c^3}\varepsilon^3 \cdot \dfrac{1}{e^{\beta\varepsilon}-1}$

This is the energy density per photon in the energy range dε, centered on energy ε.

We now calculate $u = \int du$, the total energy of the photons of all energies, for a system at temperature T=$\dfrac{1}{\beta k}$.

$$u = \int du = \frac{8\pi}{h^3c^3}\int_0^\infty \frac{\varepsilon^3 d\varepsilon}{e^{\beta\varepsilon}-1}$$

Let $x = \beta\varepsilon$, so $\varepsilon^3 = \dfrac{x^3}{\beta^3} = k^3T^3x^3$ and $d\varepsilon = \dfrac{dx}{\beta} = kTdx$

$$u = \frac{8\pi\left(kT\right)^4}{h^3c^3}\int_0^\infty \frac{x^3dx}{e^x-1} = \frac{8\pi\left(kT\right)^4}{h^3c^3}\cdot\frac{\pi^4}{15} = \left(\frac{8\pi^5k^4}{15h^3c^3}\right)T^4$$

$u = \text{constant}\cdot T^4$ This u is the energy density of the photon gas.

Example application:

Suppose the sun has radius $r = 8\times10^{8}\,m$. The sun's volume $V = \dfrac{4}{3}\pi r^{3}$.

The sun's average temperature = $3\times10^{6}\,K$.

If all of the sun's energy is stored as a photon gas in its interior, how much energy is in the sun?

$$u = \frac{\text{energy}}{\text{volume}} = \left(\frac{8\pi^{5}k^{4}}{15h^{3}c^{3}}\right)T^{4}$$

$$\text{So energy} = u\cdot\text{Volume} = \left(\frac{8\pi^{5}k^{4}}{15h^{3}c^{3}}\right)T^{4}V$$

$$= \frac{8\pi^{5}\cdot\left(1.381\times10^{-23}\,\dfrac{J}{K}\right)^{4}\cdot\left(3\times10^{6}\,K\right)^{4}\cdot\dfrac{4}{3}\pi\cdot\left(8\times10^{8}\,m\right)^{3}}{15\cdot\left(6.63\times10^{-34}\,Js\right)^{3}\left(3\times10^{8}\,\dfrac{m}{s}\right)^{3}}$$

$$= 8.8\times10^{37}\,J$$

The relation, $u \propto T^4$, is a consequence of the three-dimensionality of space.

We now show this:

At temperature T, a photon has energy of the order kT : $\varepsilon \approx kT$. (k = Boltzmann's const)

But from relativity: $\varepsilon = pc$

and from quantum mechanics, $p = \hbar K$. (this K is wave number)

So $\varepsilon = \hbar K c$.

Thus $\hbar K c = kT$

So a typical photon at temperature T has wave number $K = \dfrac{kT}{\hbar c}$.

Recall # states = $\left(\text{density of states}\right) \times \left(\text{volume of phase space being examined}\right)$

So # photons = $g(\varepsilon) \times \left(\dfrac{V_r V_p}{h^3} \right)$

But this $V_p \propto V_K$

So # photons $\propto V_K \propto K^3 = \left(\dfrac{kT}{\hbar c} \right)^3$

Photon energy density u = $\left(\text{\# photons}\right) \times \left(\text{typical photon energy}\right)$

$= \left(\dfrac{kT}{\hbar c} \right)^3 \times kT = \text{constant} \cdot T^4$

II. The Stefan-Boltzmann Law

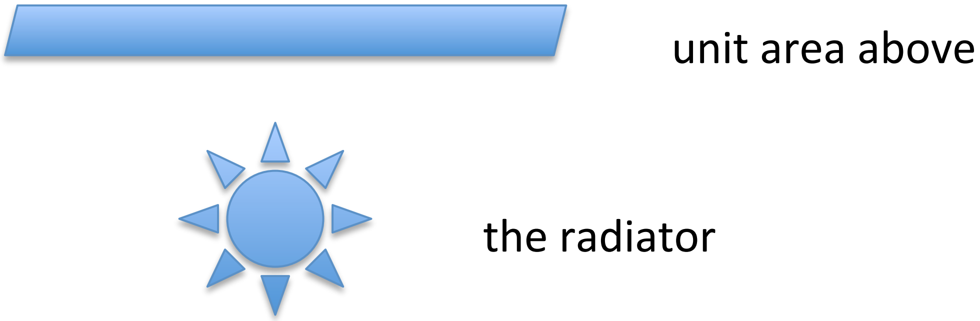
Recall the flux of anything = rate of incidence upon a unit area.

Recall for general gases (Chapter 19), $f_x = \rho \overline{v_x}$.

Write this for the z-direction as $f_z = \rho \overline{v_z}$.

Our goal: find the energy flux radiated by a body at temperature T.

Suppose the unit area is oriented in the x-y plane, so the flux will be in ẑ.



unit area above

the radiator

We want to find f_z. We know:

$\rho =$ energy density approaching the unit area

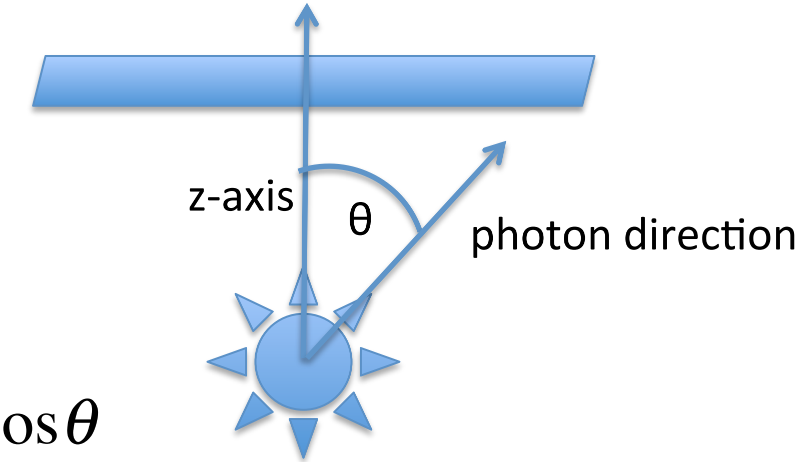
$u =$ total energy density

½ of the photons go the wrong way (-ẑ) so only $\frac{u}{2}$ contributes to the f_z.

Thus $\rho = \frac{u}{2}$.

To find $\overline{v_z}$:

All the photons travel at speed $v = c$, but their v_z component depends on angle θ of emission with respect to the z-axis.



$$v_z = c\cos\theta$$

To find $\overline{\mathrm{v}_z}$, average over $0 \le \phi \le 2\pi$ and $0 \le \theta \le \dfrac{\pi}{2}$

(all directions that contribute to upward flux).

$$\overline{v_z} = c \iint_{\theta,\phi} \cos\theta \cdot \sin\theta \, d\theta \, d\phi = \frac{c}{2}.$$

$$\text{So flux=}\left(\text{density}\right)\times\left(\text{average velocity}\right) = \frac{u}{2}\cdot\frac{c}{2} = \frac{uc}{4}$$

$$\text{Recall } u = \left(\frac{8\pi^5 k^4}{15h^3c^3}\right)T^4$$

$$\text{So flux } = \frac{1}{4}\cdot\left(\frac{8\pi^5 k^4}{15h^3c^3}\right)T^4 \cdot c$$

$$= \left(\frac{2\pi^5 k^4}{15h^3c^2}\right)T^4$$

Call this σ, the Stefan-Boltzmann constant

Then flux $= \sigma T^4$. This is the Stefan-Boltzmann Law.

III. Reflection, absorption, and emission by thermal bodies

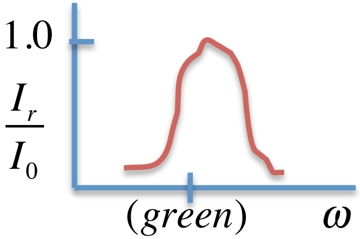
A: some facts about photons that are incident upon a thermal body:

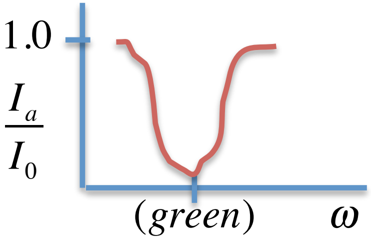
1) For any interaction of photons and a body, energy is conserved.
So a photon cannot just go out of existence without its energy being accounted for.
If I_0 = intensity of incident photons
I_r = intensity of reflected photons
I_a = intensity of absorbed photons
Then $I_0 = I_r + I_a$.

2) Whether they are absorbed or reflected depends upon the match between photon wavelength and body reflectivity.
(i) A body that appears as color X reflects wavelength X and absorbs all else.
(ii) A "blackbody" reflects no photons, absorbs all.

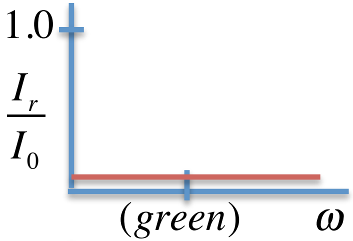
At every wavelength present, $I_r + I_a = I_0$, i.e., $\dfrac{I_r}{I_0} + \dfrac{I_a}{I_0} = 1$. Examples:

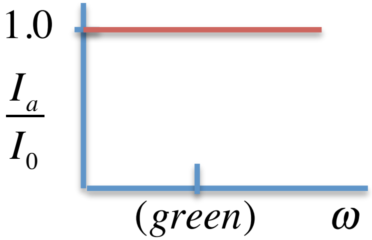
white light, green body



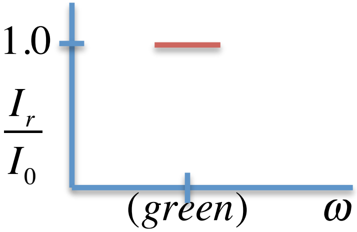


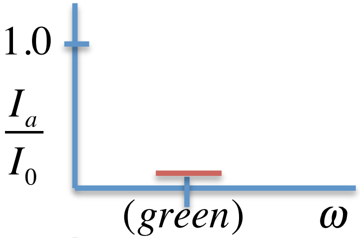
white light, black body



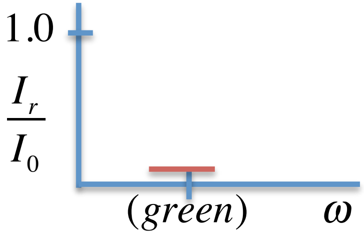


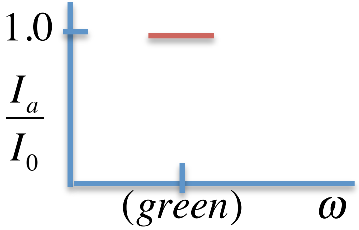
green light, green body



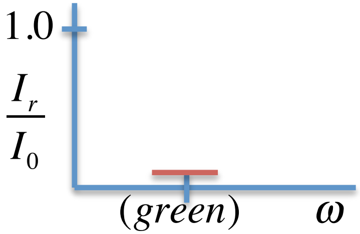


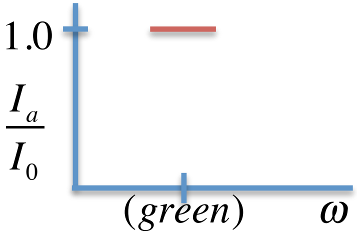
green light, red body





green light, black body

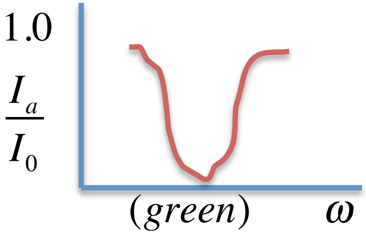


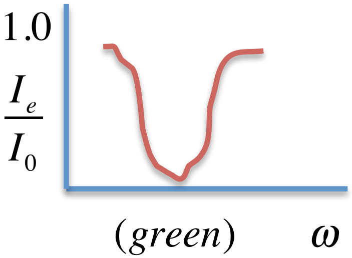


B: Facts about photon emission from bodies.

1) Let I_e = intensity of light emitted by a body. If the body is in equilibrium with the photon gas (i.e. radiation field), then it can not undergo net gain or loss of energy, so $I_e = I_a$.

For example, "$I_e = I_a$" means that the areas under these 2 curves must be the same:



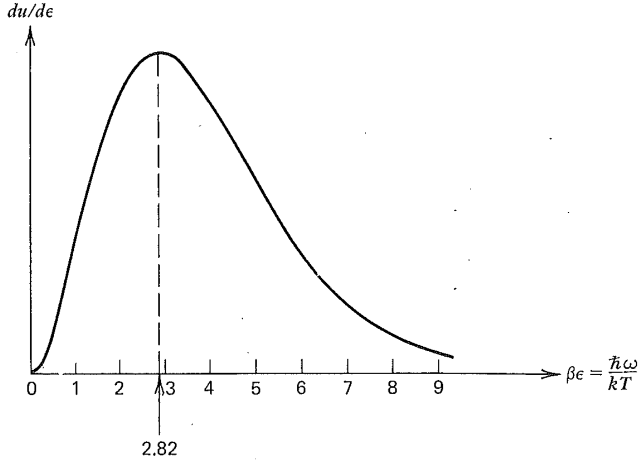


Could it happen that the curves might have different shapes but the same area?

No, because this would imply that different frequencies are emitted than absorbed, which would not conserve energy.

Thus: $I_e = I_a$ **at every frequency.** This is the Principle of Detailed Balance.

2) Recall that a blackbody absorbs all frequencies and emits all frequencies. Consider a blackbody in equilibrium with radiation at temperature T. The way to ensure equilibrium is to enclose both the body and the radiation in some vessel at temperature T. The radiation in the enclosure will have the spectrum



because this is characteristic of any body at T.

The enclosed body is bombarded by photons with this spectrum, and since it is black, it absorbs them all with this spectrum. Then by the Principle of Detailed Balance, it must emit this spectrum as well.

So this spectrum is called **the blackbody radiation curve.**