\[ H = \frac{\dot{r}^2}{2\mu} + \frac{1}{2} \mu \omega^2 r^2 = H_x + H_y + H_z \]

So solutions factorize in Cartesian coordinates:

\[ \Psi(r^2) = \Phi(r) \Theta(\theta) \Phi(\phi) \]

\[ E = (n_x + n_y + n_z + \frac{3}{2})\hbar \omega \]

To understand shell structure we need spherically symmetric solutions:

\[ \Psi(r^2) = \frac{U(\phi)}{\nu} Y_{m}(\theta, \phi) \]

Define dimensionless variables

\[ \xi = \frac{\mu \omega}{\hbar} r \quad \chi = \frac{2E}{\hbar \omega} \]

to get radial equation

\[ U'' - \frac{(\ell + 1)}{\xi^2} U' - \xi^2 U = -2U \]

at small \( \xi \)

\[ U \sim \frac{\xi^{\ell+1}}{\ell + 1} \quad \xi \rightarrow 0 \]

at large \( \xi \)

\[ U \sim \xi^2 e^{-\xi^2/2} \quad \xi \rightarrow \infty \]

Substitute

\[ U = \xi^{\ell+1} e^{-\xi^2/2} \Phi(\xi) \]
lec 8

\[ u' = \left[ (e+1)^{e_2} - 2^{e_2} \right] e^{-e_2} f + g e^{-e_2} f' \]

\[ = g^{e_2} \left( e^{-e_2} \right) f + g' \]

\[ u'' = g^{e_2} \left( e^{-e_2} \right) \left( \frac{e+1}{e} - 2 \right) f + 2 g' \left( \frac{e+1}{e} - 2 \right) + g'' \left( \frac{e+1}{e} - 2 \right) \]

\[ = f + \left( \frac{e+1}{e} - 1 \right) \]

lead to

\[ (e-2e+3)f + 2f' \left( \frac{e+1}{e} - 2 \right) + f'' = 0 \]

power series solution let \( f = \sum_{n=0}^{\infty} C_n e^{n} \)

\[ f' = \sum_{n=0}^{\infty} (n+1) C_{n+1} e^{n} \]

\[ f'' = \sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} e^{n} \]

\[ I \left( \lambda - 2e - 3 \right) C_n e^{n} + 2(n+1) C_{n+1} e^{n} \left( \frac{e+1}{e} - 2 \right) \]

\[ + (n+2)(n+1) C_{n+2} e^{n} \]

\[ f = 0 \]

term with \( \frac{e+1}{e} \):

\[ \sum_{n'=1}^{\infty} 2(n'+2)(n'+1) C_{n'+2} e^{n'} \]

\[ = \sum_{n'=1}^{\infty} 2(n'+2)(n'+1) C_{n'+2} \]

\[ n' = n-1 \]
continuing with $\frac{e^{-1}}{s}$ term

$$\frac{2}{s}e^{-1} + \sum_{n=0}^{\infty} 2(2n+2)(n+1) C_{n+2}$$

- $s$ term:

$$\sum_{n=0}^{\infty} \frac{2(2n+2)}{n+1} C_{n+2} = -\sum_{n=0}^{\infty} 2n C_n g^n$$

$$- \sum_{n=0}^{\infty} 2n C_n g^n$$

then we have

$$\frac{2}{s}e^{-1} + \sum_{n=0}^{\infty} \left( (n+2)(n+1)^2 C_{n+2} - 2n C_n \right)$$

$$\left[ 2(n+2)(n+1) + (n+2)(n+1) \right] C_{n+2} = 0$$

Since $g$ cannot be eliminated from $C_1$ term,

$$C_1 = 0.$$ Then

$$\sum_{n=0}^{\infty} \left( (n+2)(n+3) C_n + (n+2)(n+1) \right) C_{n+2} = 0$$

correcting $C_{n+2}, C_n$. $C_1 = 0$ implies all odd coefficients are zero.
Let $s$

Recursion relation is

\[
\frac{c_{n+2}}{c_n} = \frac{2n+2l + 2}{(n+2)(n+3+2l)} \to \frac{2}{n+1}
\]

At large $n$, this is the same as expansion of

\[
\delta^2 = \sum_{\text{even}} \frac{s_{\frac{n}{2}}}{(\frac{n}{2})!} s_{\frac{n}{2}} - \frac{2}{\text{reursion } n+1}
\]

Some must terminate and

\[
\lambda = 2(n+1+3/2) = \frac{2E}{\hbar} \quad n = 0, 1, 2, \ldots
\]

or $n = 2n_\perp, \ n = 0, 1, 2 \ (\neq \text{radial node})$

Sketch of radial wave function:

![Sketch of radial wave function](image-url)
On the you find that states with same \( L \) are degenerate. Just as for 1/2 potential r^2 potential has classically closed orbits and quantum dynamical symmetry.

You find closed shells 2, 8, 20, 40 giving first 3 magic numbers.

To do better add spin-orbit term.

Nuclei with closed shells have \( \tilde{J} = 0 \) (\( \frac{A}{2} N \) notation)

\[
16, 40, 80, 208
\]

Nuclei with \( n \) nucleon nuclei have \( \tilde{J} = \frac{1}{2} \)

\[
15_N, 39_{K}, 207_Pb
\]