Magic numbers & Shell Model

1948 Maria Goeppert Mayer (Nobel 1963)

Certain values of \( Z \) or \( N \) (neutron = \( A - Z \)) are particularly stable ("magic")

\[ 2, 8, 12, 18, 28, 50, 82, 126 \]

Some are doubly magic \( 4, 8, 8 \)

Suggests shell structure with filled shells particularly stable

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**Fig. 3.6** Observed neutron separation energies \( (S_n)_\text{exp} \) compared with \( (S_n)_\text{cal} \) predicted by the smooth variation of the semiempirical mass formula, using Fermi 1945 coefficients (Table 3.3) in Eq. (3.65). Discontinuities of the order of 2 Mev are evident for \( N = 50, 82, \) and 126. Evidence for a shell closure at \( N = 28 \) is inconclusive.

[Adapted from Harvey (122).]

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Neutron separation energies compared to smooth variation predicted by semiempirical mass formula.

Single particle model \( (N = \text{nucleon, p,n,n}) \)

\( N-N \) interaction in nuclear ground state \( (\text{filled shells}) \)

nearly absent due to exchange symmetry.

Assume remaining \( N \) move in effective potential of other nucleons.

Fermi-Geer model + estimate of depth of nuclear well

\[
\begin{align*}
E & \quad \uparrow \quad \downarrow \\
E_F & \quad \text{Contents} \\
V_0 & \\
E_b + E_F & = V_0
\end{align*}
\]

\[
E_F = \frac{p_F^2}{2m} \quad \text{Fermi momentum}
\]

Number density in quantum phase space

\[
dN = V \frac{d^3p}{\hbar^2} \quad \text{where } V \text{ is nuclear volume}
\]

\[
V = \frac{4\pi}{3} r^3 = \frac{4\pi}{3} \left( \frac{A}{A_0} \right)^3 = \frac{4\pi}{3} \text{A} r_0^3
\]
For degenerate gas, states are filled to $E_F$:

\[ \int d^3p = 4\pi \int_0^{p_F} p^2 dp = \frac{4\pi}{3} p_F^3 \]

Total number for neutron or proton:

\[ N = \frac{V}{(2\pi^2)^3} \left( \frac{4\pi}{3} p_F^3 \right) \times 2 \times 4 \times \text{spin} \]

\[ = \left( \frac{1}{2\pi^2} \right)^3 \left( \frac{4\pi}{3} \right) A \left( \frac{r_0 p_F}{\hbar} \right)^3 \]

\[ = \frac{4\pi}{3\hbar} A \left( \frac{r_0 p_F}{\hbar} \right)^3 \]

For heavy nucleus take $Z \ll A - Z$:

\[ N = \frac{Z}{A} = \frac{4\pi}{9\pi} p_F^3 \left( \frac{r_0 p_F}{\hbar} \right)^3 \]

Then $p_F$ is independent of $A$:

\[ p_F = \frac{1}{r_0} \left( \frac{9\pi}{8} \right)^{1/3} \approx 1.17 \text{ MeVfm} \]

\[ E_F = \frac{p_F^2}{2m_n^2} = \frac{1}{2m_n^2} \left( \frac{hc}{r_0} \right) \left( \frac{9\pi}{8} \right)^{1/3} \approx 3.3 \text{ MeV} \]

Depth of well estimates:

\[ V_0 = E_F + E_F \approx (8 + 3.3) \text{ MeV} = 41 \text{ MeV} \]
Nuclear Potential Model

\[ V(r) \]

\[ \text{harmonic oscillator} \]

\[ -40 \text{ MeV} \]

In general, \( \ell \neq 0 \).

**Infinite square well**

\[ \frac{1}{2} m E / \hbar \]

\[ \frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} - \frac{\ell (\ell + 1)}{r^2} R + k^2 R = 0 \]

\[ \hbar = \frac{12 m E / \hbar}{\text{note that for infinite well, } E > 0.} \]

Define \( s = kr \), \( R' = dR/ds \)

\[ R'' + \frac{2}{s} R' + \left( 1 - \frac{\ell (\ell + 1)}{s^2} \right) R = 0 \]

Semi-solution yields spherical Bessel function

\[ j_{\ell}(s) = (-1)^{\ell} \frac{1}{\sqrt{s}} \frac{d^{\ell} \sin s}{ds^\ell} \]

And Neuman functions \( n_{\ell}(s) \) but not finite at \( s = 0 \) and therefore do not apply.
Spherical Bessel functions:

\[ j_0(s) = \frac{\sin s}{s} \]

for \( s \to \infty \)

\[ j_m(s) \sim \frac{s^m}{\Gamma(m+1)} \]

\[ j_0(s) \sim \frac{1}{s} \cos \left( s - \frac{\pi}{2} (2m+1) \right) \]

These functions also describe diffraction through circular apertures.

Sketch:

\[
\begin{align*}
&1.0 \quad j_0 \quad j_1 \\
&1.2 \quad 8 \quad s
\end{align*}
\]

Energy determined by \( j_0(ka) = 0 \) boundary condition.
for $k > 0$

$$j_0(ka) = \frac{\sin(ka)}{ka} = 0$$

Given $ka = n \pi$, $n$ also counts number of nodes in $j_0$.

$$E_{n,0} = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2\mu} \left( \frac{n \pi}{a} \right)^2$$

Energy spectrum: $(n \pi/a)^2 = \frac{2\mu E}{\hbar^2} a^2$

\[ (n \pi/a)^2 / 100 \]

- $n = 3, l = 2$
- $n = 2, l = 1$
- $n = 1, l = 0$

$\ell = 2(n + 1)$

Closed shells: 2, 8

First 2 magic numbers