Statistics IV: Hypothesis Testing

\[ -\ln L(\theta) \]

\[ \frac{1}{2} l - \ln L(\theta^*) \]

\[ \theta^* - \delta \theta \quad \theta^* \quad \theta^* + \delta \theta \]

\[ -\ln L(\theta) \equiv l(\theta) = l(\bar{\theta}) + \frac{1}{2} \left( \frac{\theta - \bar{\theta}}{\delta \bar{\theta}} \right)^2 \]

\[ \delta \bar{\theta}^{-2} = \left. \frac{d^2 l}{d \theta^2} \right|_{\bar{\theta}} \]

Measurement: \( \theta^* \pm \delta \theta^* \) estimate true \( \bar{\theta} \)

Value of \( L(\theta^*) \) tests hypothesis made on

for relation likelihood

\[ L(\theta) = \prod P(x_i; \theta) \]

For example, \( P(x_i; \theta) = \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left( -\frac{1}{2} \left( \frac{y_i - \theta x_i}{\sigma^2} \right)^2 \right) \)

is a test of the linear hypothesis \( y_i = \theta x_i \)
The values $\theta^*$ are distributed about $\theta$ according to a universal, calculable PDF that depends only on the number of free random variables (degree of freedom) in the likelihood.

$$\chi^2 = 2 \ln (l(\theta^*)) = -2 \ln l(\theta^*)$$

$$ndf = [\# \text{ data points}] - [\# \text{ fitted parameters}]$$

The chi-square distribution (PDF) $f_{\chi^2}(\chi^2)$

$$f_{\chi^2}(\chi^2)$$

$$\langle \chi^2 \rangle = ndf$$

$$\sigma_{\chi^2} = 2 \sqrt{ndf}$$

most probable $ndf \gg 1$, $\chi \sim ndf$
As a measure of goodness of fit, we calculate the probability to get

\[ \chi^2 > \chi^2(\alpha) \]

measured value

\[ P_{\text{val}} = \int_{\chi^2(\alpha)}^{\infty} f_{\chi^2}(x^2) \, dx^2 = 1 - CDF \]

where CDF = cumulative distribution function

Note: web page CDF calculator

Greater P means better fit.

"Rule of thumb": reduced \( \chi^2 \leq 1 \) for good fit.

\[ \frac{\chi^2(\alpha)}{\nu} \times 1 \]
Figure 32.1: One minus the χ² cumulative distribution, 1 - F(χ²; n), for n degrees of freedom. This gives the p-value for the χ² goodness-of-fit test as well as one minus the coverage probability for confidence regions (see Sec. 32.3.2.3).
Curves show as a function of $n/u$ that corresponds to a given $p$-value. Figure 32.7: The reduced $X^2/n$ equal to $X^2$ for $n$ degrees of freedom. The degrees of freedom in

![Graph of Chi-square distribution]
Unknown error \( \sigma_y \).

We cannot calculate a \( p \)-value. However, if we assume \( \sigma_y^2 = \text{constant} = \sigma_x^2 \),

we can make a best estimate of \( \sigma_x \) by setting

\[ \chi^2(\theta^*) = n \nu f. \]

\[ \chi^2(\theta^*) = n \nu f = \frac{1}{\sigma_x^2} \sum (y_i - y(\theta^*, x_i))^2 \]

\[ \sigma_x^2 = \frac{1}{n \nu f} \sum (y_i - y(\theta^*, x_i)) \]

called "point-error estimate" or \( \sigma \)-factor.

**Example:** delay vs. ADC counts

![Graph showing delay vs. ADC counts with a linear fit.](attachment:graph.png)

**Calibration:** \( m \pm \sigma_m \) from fit

**Estimated time resolution (point-error estimate)** \( \sigma_x \)