Mutual Inductance:

Consider coaxial coils: "primary" $N_1, i(t)$  
"secondary" $N_2, i_2(t)$

$E_{mf}$ in secondary given time-dependent current $i(t)$ in primary.

Apply Lenz's law:

$E_2 = -N_2 \frac{d\Phi_2}{dt}$

Where flux through single turn of secondary

$\Phi_2 = \int B_1 d\mathcal{A} = B_1\pi r_1^2$

$B_1 = \mu_0 \frac{N_1}{l_1} i_1$

$\Phi_2 = \pi r_1^2 \left( \mu_0 \frac{N_1}{l_1} \right) \frac{di_1}{dt}$

$E_2 = -\mu_0 N_1 N_2 \left( \frac{\pi r_1^2}{l_1} \right) \frac{di_1}{dt} = -M_{21} \frac{di_1}{dt}$
Coefficient $M_{21}$ is the mutual inductance measured in Henrys:

$$V = IR, \quad P = I^2R$$

$$N_2 = \frac{V}{(A/s)} = A \cdot S = \frac{J}{A^2}$$

\[\text{(2) Reciprocity theorem}\]

\[M_2 = M_{21}\]

\[\Phi_1 = \int B_2 \cdot dA = B_2 \pi r_i^2 = \left(\mu_0 \frac{N_2}{L_2}\right) \pi r_i^2\]

\[\phi_1 = -\left(\frac{M_1}{\ell_1}\right) l_2 \Phi_1 = -\mu_0 N_1 N_2 \frac{\pi r_i^2 \, dz}{\ell_1}\]

\[M_2 = \mu_0 \frac{N_1 N_2}{L_1} \pi r_i^2 = M_{21}\]

Can prove that this is always true.
Transformer

Magnetic field stays inside iron core -

\[ \Phi_{B_1} = \Phi_{B_2} \]

\[ E_1 = -N_1 \frac{\Delta \Phi}{\Delta t} \]
\[ E_2 = -N_2 \frac{\Delta \Phi}{\Delta t} \]

\[ \frac{E_1}{E_2} = \frac{N_1}{N_2} \Rightarrow E_2 = \left( \frac{N_2}{N_1} \right) E_1 \]

Step-up (step-down) Voltage with time-varying (AC) current.
Single coil self-inductance gives "back emf" 

\[ \mathcal{E} = -N \frac{d\Phi}{dt} = -L \frac{dI}{dt} \]

\[ \Phi = B \pi r^2 = \mu_0 (N/2) \pi r^2 \]

\[ L = \frac{\mu_0 N^2 \pi r^2}{l} \]

\[ \Rightarrow n = \frac{N}{2} \text{ turns/length} \]

Coil has inductance per unit length, 

\[ \frac{L}{l} = \mu_0 n^2 \pi r^2 \]
In the circuit shown:

\[ E = L \frac{dI}{dt} - IR = 0 \]

This is the equation for the inductor, where \( E \) is the emf, \( I \) is the current, \( L \) is the inductance, \( R \) is the resistance, and \( \frac{dI}{dt} \) is the rate of change of current.

A mmf for back emf:

\[ \frac{E}{R} = \frac{L}{R} I - I = 0 \]

Solution:

\[ I(t) = A e^{-t/\tau} + B \]

At \( t=0 \), \( I=0 \) \( \Rightarrow \) \( I(0) = C (1 - e^{-t/\tau}) \)

At \( t=\infty \), \( I=0 \) \( \Rightarrow \) \( I(\infty) = \frac{E}{R} (1 - e^{-\infty}) \)

Current cannot start instantaneously due to self-inductance of coil.
After many $\times \frac{1}{R}$, $I = \frac{E}{R}$, then open switch:

$$-L \frac{dI}{dt} - IR = 0$$

$$I(t) = \frac{E}{R} e^{-t/R}$$

Current in circuit decay exponentially.

Where does energy for this current come from?

 Stored in B-fields of inductor.

$$\frac{dU_B}{dt} = P = I^2 R = -L I \frac{dI}{dt} = \frac{L}{2} \frac{d}{dt} I^2$$

$$\Delta U_B > 0$$

Integrate, $$U_B = \frac{L}{2} I^2$$ stored magnetic energy

Consider long solenoid:$$L = \mu_0 n^2 \pi r^2$$

$$\frac{U_B}{\text{Volume}} = \frac{1}{V} \frac{L}{2} I^2 = \frac{1}{2} \mu_0 n^2 I^2$$

$$B = \mu_0 n I \Rightarrow \mu_B = \frac{1}{2} \frac{1}{\mu_0} B^2$$

Recall $$E_x = \frac{E^2}{2}$$
LC circuit

\[ V = iL \]

\[ \frac{d}{dt} \left( \frac{Q}{L} \right) = \frac{Q}{LC} \]

\[ 2 \pi \omega = \sqrt{\frac{1}{LC}} \]

Choose \( i = \frac{dQ}{dt} \)

Kirchhoff's loop rule

\[ -\frac{Q}{C} - L \frac{dQ}{dt} = 0 \]

\[ \frac{Q}{LC} = \phi(t) \]

\[ \phi(t) = A \cos(\omega t) + B \sin(\omega t) \]

where \( \omega = \frac{1}{\sqrt{LC}} \) angular frequency

\[ \omega = 2\pi \frac{1}{T} \]

\[ \dot{\phi}(t) = -\omega \phi \sin(\omega t + \phi) \]

Current oscillates with period \( T = \frac{2\pi}{\sqrt{LC}} \)

Analogous to mass on spring with \( L \leftrightarrow \text{mass} \) and \( \frac{1}{C} \leftrightarrow \text{k} \)

\[ \omega = \frac{1}{\sqrt{LC} \left( \frac{k}{m} \right)} \]
The phase rule of oscillator coordinate, $\dot{\phi} = \dot{\theta}$, is analogous to velocity.

Magnetic energy $U_B = \frac{1}{2} \dot{\theta}^2 \left( \frac{1}{2} m v^2 \right)$

Electric-field energy $U_E = \frac{1}{2} \frac{Q^2}{c^2} \left( \frac{1}{2} k x^2 \right)$

$U_B + U_E = \frac{1}{2} L \omega^2 Q^2 \sin^2 \theta + \frac{1}{2} \frac{Q^2}{c^2} c_0^2$

$= \frac{1}{2} \frac{Q^2}{c^2}$ constant with time.

$\dot{\theta} = \dot{\phi} = -\omega \varphi \sin \theta = \pm \omega \varphi \sqrt{1-c_0^2}$

$= \pm \omega \sqrt{\varphi^2 - \delta^2} \quad (v_x = \frac{\sqrt{B}}{m} \sqrt{A^2 - x^2})$
Example 3.0.75

\[ \begin{array}{c}
\text{Example 3.0.75} \\
\begin{array}{c}
E \\
S \\
\frac{1}{2} R_1 \\
\frac{1}{2} R_2 \\
R_2 \\
\end{array}
\end{array} \]

Close switch at \( t = 0 \).

\[ \begin{align*}
E - i_1 R_1 &= 0 \\
E - L \frac{di_2}{dt} - i_2 R_2 &= 0
\end{align*} \]

\[ i_1 = \frac{E}{R_1} \]

\[ i_2 = \frac{E}{R_2} \left( 1 - e^{-\frac{R_2}{L} t} \right) \]

\[ P_1 = i_1^2 R_1 = \frac{E^2}{R_1} \]

After time \( t >> \frac{L}{R_1} \), switch is opened. Then

\[ -i R_1 - i R_2 - L \frac{di}{dt} = 0 \]

where \( i(0) = \frac{E}{R_2} \)

\[ \frac{di}{dt} = -\left( \frac{1}{R_1 + R_2} \right) i \]

\[ i = \frac{E}{R_2} e^{-\frac{R_1 + R_2}{L} t} \]

Power in \( R_1 \) immediately after opening switch:

\[ P_1(t) = i^2 R_1 = \frac{E^2}{R_2^2} \cdot R_1 \]

\[ \frac{P_1(t)}{P_1} = \frac{R_1}{R_2^2} \]
L-C-R series circuit

Damped oscillator

\[
\frac{d^2 q}{dt^2} + \frac{Q}{2} \frac{dQ}{dt} + \frac{Q}{LC} = 0
\]

Two constants with dimensions of time -

\[
\sqrt{LC} = \frac{1}{\omega}, \quad \frac{1}{R} = \tau
\]

3 types of damping:

under damped
over damped
critically damped

Try

\[
Q(+) = e^{-at} f(+)
\]

\[
\dot{Q} = -a e^{-at} f + e^{-at} \dot{f}
\]

\[
\ddot{Q} = a^2 e^{-at} f - 2ae^{-at} \dot{f} + e^{-at} \ddot{f}
\]

Substitute into

\[
\ddot{Q} + \frac{Q}{C} + \omega^2 Q = 0
\]
overall factor of $e^{-at}$ cancels giving

$$\omega^2 \ddot{f} + \frac{1}{\varepsilon} (-af + \dot{f}) + a^2 f - 2a \ddot{f} + f = 0$$

$$\ddot{f} + \dot{f} \left( \frac{-1}{\varepsilon} - 2a \right) + \left( \omega^2 - \frac{a}{\varepsilon} + a^2 \right) f = 0$$

$$= 0 \quad \varepsilon \quad a = \frac{1}{2\varepsilon}$$

$$\omega^2 - \frac{a}{\varepsilon} + a^2 = \omega^2 - \frac{1}{2\varepsilon^2} + \frac{1}{4\varepsilon^2} = \omega^2 - \frac{1}{4\varepsilon^2} \equiv \omega'^2$$

Then

$$\ddot{f} = -\omega'^2 f$$

$$\Rightarrow f = Q \cos (\omega' t + \phi)$$

$$\theta(t) = Q e^{-\frac{1}{2} \xi t} \cos (\omega' t + \phi)$$

max. capacitor charge

phase

This is underdamped solution - $\omega'^2 > 0$

$$\omega^2 - \frac{1}{4\varepsilon^2} > 0$$

Critical damping - $\omega^2 - \frac{1}{4\varepsilon^2} = 0$

Over damped - $\omega^2 - \frac{1}{4\varepsilon^2} < 0$

$$\ddot{f} = \left( \frac{1}{4\varepsilon^2} - \omega^2 \right) f$$

$$1/2\varepsilon^2$$
Overdamped: \( q(t) = Q e^{-\frac{1}{4 \tau^2}} e^{-\frac{t}{\tau}} \)

Exponential decay without oscillation

For small damping:

\[ \omega^2 \gg \frac{1}{4 \tau^2}, \quad \frac{1}{\zeta} \gg \frac{1}{4 \zeta^2} \]

\[ \sqrt{\zeta} \ll \frac{2}{L/R} \]

\( \omega' \approx \omega \)

\[ q(t) = Q e^{-\frac{t}{\tau}} \cos \omega' t \]

Charge on capacitor versus time.