Capacitance

I. Capacitor
A device for storing charge and energy in an electrical circuit. 2 conductors, separated in space with a potential difference across them.

Simplest geometry is 2 plates (separated) ignore non-uniform "fringe" field near edges.

\[ V(x) = V_0 - E \cdot x \]

\[ V(0) = V_0 \]

\[ \nabla \phi = \frac{V_0}{d} \]

\[ E = \frac{V_0}{d} \]
Surface charge: Gauss's surface box of side $a$ and enclosing region of plate at $x = d$.

\[
\int \mathbf{E} \cdot d\mathbf{S} = -E x = \frac{\sigma a}{\varepsilon_0}
\]

Similarly,

\[
\sigma_{x=0} = \frac{V_0}{d} \varepsilon_0
\]

Change on plate $\pm Q$ with

\[
Q = \sigma \varepsilon \left( \frac{x \varepsilon_0}{d} \right) V_0 = C V_0
\]

Debye's capacitance. Constant $C$ depends only on geometry of conductors and material in gap. Here, gap is vacuum.

Capacitance $C_{ii} = \frac{a \varepsilon_0}{d}$
### Capacitor Stores Energy

\[ \begin{align*}
+Q & \quad -Q \\
\vdots & \quad \vdots \\
\Delta & \quad \Delta x \\
\epsilon & \quad \epsilon
\end{align*} \]

Move charge \( \Delta Q \) from right side and transfer to left.

**External Work** = \( \Delta Q \cdot Ed = +\Delta U \)

Here it \( w \) is \( +\Delta U \) because it is external work done against the field.

\[ E(Q) = \frac{\phi}{\epsilon_0} = \frac{\phi}{\epsilon_0} \]

\[ \Delta Q \cdot Q \left( \frac{d}{\epsilon_0} \right) = \Delta U \]

\[ \frac{du}{dQ} = \frac{Q}{C} \]

Start from \( Q = 0 \) and integrate:

\[ U(Q) = \int Q \frac{\phi d\phi}{C} = \frac{1}{2} \frac{\phi^2}{C} \]

\[ U_{stored} = \frac{1}{2} \frac{\phi^2}{C} = \frac{1}{2} CV_0^2 \]
we can also rewrite in terms of $E^2$ between plates:

$$U_c = \frac{1}{2} c (dV_0)^2 = \frac{1}{2} \left( \frac{\varepsilon_0 a}{d} \right) d^2 E^2$$

$$U_c = \frac{1}{2} \varepsilon_0 (a d) E^2$$

- Volume of capacitor.

*physical interpretation: Energy stored in $E$ field which has density/Volume

$$\frac{U}{V} = \mu = \frac{1}{2} \varepsilon_0 E^2$$

*Wow!*
Example: Self energy of sphere of radius a, charge Q:

\[ U = \frac{1}{2} k \frac{Q^2}{a} \]

\[ E(r) = \frac{kQ}{r^2} \quad r > a \]

\[ U = \int d^3r \cdot \frac{1}{2} \varepsilon_0 E^2(r) \]

\[ = \frac{4\pi \varepsilon_0}{2} \int_a^\infty dr \left( \frac{kQ}{r^2} \right)^2 \]

\[ k = \frac{1}{4\pi \varepsilon_0} = \frac{kQ^2}{2} \int_a^\infty dr \frac{dr}{r^2} = \frac{1}{2} k \frac{Q^2}{a} \]

\{ Interpretation of field energy density is consistent. \}
Example: Force between capacitor plates

\[ C_{11} = \frac{\varepsilon_0}{x} \quad \text{separation } x \]

\[ \frac{dC}{dx} = -\frac{\varepsilon_0}{x^2} = -\frac{C}{x} \]

\[ U = \frac{1}{2} \frac{q^2}{C} \]

\[ F_x = -\frac{dU}{dx} = -\frac{1}{2} \frac{q^2}{C^2} \left( -\frac{1}{x^2} \right) \left( -\frac{C}{x} \right) = -\frac{U}{x} \]

Attractive force

\[ U = \frac{1}{2} C V^2 \]

\[ F_x = -\frac{dU}{dx} = +\frac{U}{x} \]

Repetitive force
Example: Spherical Capacitor

\[ C = \frac{Q}{V} \]

\[ V(b) - V(a) = -\int_a^b E \cdot dr = -\int_a^b \frac{kQ}{r^2} \, dr \]

\[ = kQ \left( \frac{1}{b} - \frac{1}{a} \right) \]

define \( V(b) = 0 \),

\[ V = V(a) = kQ \left( \frac{1}{a} - \frac{1}{b} \right) \]

\[ C = \frac{4\pi E_0 \left( \frac{ab}{b-a} \right)}{\varepsilon} \]

Capacitance is always \( \propto E_0 \times \text{(length)} \)

Take \( \lim b \to \infty \), \( V = \frac{Q}{4\pi \varepsilon_0 a} \)

\[ C = \frac{4\pi\varepsilon_0 a}{\varepsilon} \quad \text{Capacitance of single conductor.} \]
Capacitante și circuituri

Circuit element symbol

\[ V = V_1 + V_2 \]

Energy conservation

Charge conservation

\[ q_1 = q_2 = q \]

\[ V = \frac{q}{C_1} + \frac{q}{C_2} = q \left( \frac{1}{C_1} + \frac{1}{C_2} \right) \]

\[ \frac{q}{V} = \text{Coeff} = \frac{1}{C_1} + \frac{1}{C_2} \]

\[ V_{1} = V_{2} = V \]

\[ q = q_1 + q_2 = (C_1 + C_2) V \]

\[ \frac{q}{V} = \text{Coeff} = C_1 + C_2 \]

Think of 2 infinite plate capacitors with separation:

\[ C_{\text{eff}} = \varepsilon_0 \frac{q^2}{d} + \frac{\varepsilon_0 q^2}{d} = \frac{q^2}{d} (\varepsilon_1 + \varepsilon_2) \]
Dielectrics:

Fill the space between conductors with an insulating material and measure its capacitance. Compare its value with space a vacuum $(C_0)$.

\[ \frac{C}{C_0} = k \]

dielectric constant of material

<table>
<thead>
<tr>
<th>Material</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vacuum</td>
<td>1</td>
</tr>
<tr>
<td>air (1 atm)</td>
<td>1.00054</td>
</tr>
<tr>
<td>mica</td>
<td>3-6</td>
</tr>
<tr>
<td>glass</td>
<td>5-10</td>
</tr>
<tr>
<td>water</td>
<td>$\sim 81$ (T dependent)</td>
</tr>
<tr>
<td>strontium titanate</td>
<td></td>
</tr>
</tbody>
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Microscopically:

1. Molecules get induced dipole moment in presence of $E$.

2. Molecules with permanent dipole moments.

\[ \vec{E} = 0 \quad \text{or} \quad \vec{E} \]

- Distorted molecule
- Partially aligned dipoles
Molecular dipole moment measured in Debye, \( D = 3.355 \times 10^{-30} \text{ C.m} \)

atomic charge separation

\[
\text{length} = \frac{p}{e} = \frac{3.355 \times 10^{-30} \text{ C.m}}{1.6 \times 10^{-19} \text{ C}} = 2.10 \times 10^{-11} \text{ m}
\]

\[ \approx \frac{1}{5} \text{ Å} \]

\[ P(\text{H}_2\text{O}) = 1.8 \text{ D} \]

\[ P(\text{NaCl}) = 7.0 \text{ D} \]

\[ E \] between capacitor plate, reduced relative to field in vacuum

\[ +q \quad -q \quad +q \quad -q \]

\[ = \quad +q \quad -q \]

polarized dielectric slab

\[ E = E_0 + E' \]

\[ = \frac{1}{K} E_0 (E' \propto E_0) \text{ linear} \]

\[ q = \sigma \cdot a \]

\[ \frac{q}{V} = E \cdot d = \frac{1}{K} \frac{\sigma}{E_0} \cdot d \]

\[ C = \frac{q}{V} = K \left( \frac{E_0 \cdot a}{d} \right) = K C_0 \]
free and bound charge

charge separation inside dielectric does not move, but rather remains bound in material.
We rewrite Gauss's law in terms of free charge.

\[ E^2 = \frac{1}{k} E_0 \]

Gaussian surface

\[ E_0 = \frac{k Q}{r^2} \]

\[ \int \vec{E}_0 \cdot d\vec{a} = \frac{1}{k} \left( \frac{1}{4\pi \varepsilon_0} \right) \frac{Q}{r^2} (4\pi r^2) \]

\[ = \frac{Q}{k \varepsilon_0} \]

Define \( \varepsilon = k \varepsilon_0 \)

\[ \int \vec{E} \cdot d\vec{a} = \frac{Q}{\varepsilon} \]

\( \varepsilon \) called "permittivity"

\( \varepsilon_0 \) called permittivity of free space