Electromagnetism #3: Gauss's Law

Field line diverge as field gets weaker. Can we make this idea precise?

"Flow" of field through an area

Analogue to flow of fluid:

\[ \frac{dV}{dt} = \vec{v} \cdot \vec{a} \]

\[ \Delta V = \vec{v} \cdot \vec{a} \Delta \tau \]

\[ \frac{dV}{dt} = \vec{v} \cdot \vec{a} \]

\( \vec{a} \) oriented area

direction according to right hand rule
Electric flux for uniform (in space) \( \vec{E} \)

\[
\Phi_E = \vec{E} \cdot \vec{a}
\]

For non-uniform \( \vec{E}(\vec{r}) \) and arbitrary surface \( S' \), divide \( S' \) up into infinitesimal patches:

\[
\Phi_E = \sum E(\vec{r}) \cdot d\vec{a}
\]

Chosen for convenience

\[
\Phi_E = \int_{S'} E(\vec{r}) \cdot d\vec{a} \quad \text{total flux through } S'
\]
Flux through closed surface - orientation of area is defined so that da\^2 points outward.

Uniform \( \mathbf{E} \):

\[ \Phi_E = \Phi_1 + \Phi_2 = \mathbf{E}_a - \mathbf{E}_e = 0 \]

Non-uniform \( \mathbf{E}(r) \) and arbitrary closed volume

Equal \# lines exit as enter:

\[ \Phi_E = 0 \]

\( S \) with no charge inside

Equal \# line exit any surface enclosing \( Q \):

\[ \Phi_E \propto Q \]
Gauss's Law:
\[ \Phi_E = \oint E \cdot d\vec{a} = \frac{Q_{\text{enc}}}{\varepsilon_0} \]

arbitrary closed surface

Since this law holds for arbitrary surface, we can exploit symmetry of \( \vec{E}(r') \) to easily determine \( \vec{E}(r') \) for highly symmetric charge configurations.

Choose Gaussian surface at concentric sphere.

By symmetry,
\[ \vec{E}(P) = E(r) \hat{r} \]
\[ \Phi_E = \oint \vec{E} \cdot d\vec{a} \]

\[ d\vec{a} = r \left( r \sin \theta \, d\phi \right) (r \, d\theta) \]

\[ \vec{r} = r \left[ \cos \theta \, \hat{i} + \sin \theta \left( \cos \phi \, \hat{j} + \sin \phi \, \hat{k} \right) \right] \]

\[ \Phi_E = E(r) r^2 \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} d\phi = 4\pi r^2 E(r) \]

apply \underline{Gauss's Law}

\[ E(r) = \frac{\Phi_E}{4\pi r^2} = \frac{1}{\varepsilon_0} \frac{kq}{r^2} \]
Field of \( \infty \) uniform line charge:

Cylindrical symmetry \( \vec{E}(r^2) = E(r) \hat{z} \)

\[
\begin{align*}
\vec{E}(r^2) &= E(r) \hat{z} \\
\vec{r}^2 &= z \hat{z} + r (\cos \phi \hat{x} + \sin \phi \hat{y}) \\
&= z \hat{z} + r \hat{\phi}
\end{align*}
\]

Line charge on \( z \) axis with Gaussian Surface as concentric cylinder

\[
\Phi = \Phi_{\text{ends}} + \Phi_{\text{pipe}} = \int_{\text{circumference}} E(r) \vec{d}A
\]

\[
= 2\pi r E(r) \int_0^l \sin \phi \, d\phi \
= 2\pi r E(r) \left[ -\cos \phi \right]_0^{2\pi}
\]

\[
= 2\pi r E(r) \left( \frac{2\pi}{r} \right)
\]

By Gauss's law

\[
E(r) = \frac{\lambda}{2\pi \varepsilon_0} = 2k_0 \lambda
\]
Conductor up static charge:

\[ E = 0 \]

since charge is static by assumption \( E \) inside is zero and \( E \) on surface is always normal to surface.

\( \vec{E} \) on surface adjusts itself so that \( \vec{E} \) is normal:

\[ \vec{E} \]...

\( \vec{E} \) on surface would cause charge to move,

conducting sphere up hole a charge

\[ E = \frac{kQ}{r^2}, \rightarrow R \]

\[ \oint E \cdot d\vec{a} = 0 \] inside conductor

\[ -\sigma \cdot \text{area} = \Phi \]

"Today's ice pile proof that charge is on exterior surface"
Faraday cage:

Inside enclosed conductor with zero charge, \( E = 0 \) no matter what the static external charge is.

Typical Faraday cage in lab is ferromesh copper box.
Earnshaw's theorem:

A charge cannot be held in stable, static equilibrium by electrostatic fields only.

\[ e \quad \hat{e} \quad +e \quad -e \quad \rightarrow \hat{z} \]

- \[ e' \quad +e' \quad -e' \]

Displacement in symmetry plane \( \perp \hat{z} \) are stable, but along \( \hat{z} \) are unstable.

\[ e \quad \hat{e} \quad +e \quad \rightarrow \hat{z} \]

Displacement along \( \hat{z} \) are stable, along \( \hat{e} \) are unstable.

\[ \vec{E} \rightarrow \text{gives stable configuration must be:} \]

\[ \vec{E} \quad \hat{e} \]

Implying a charge \(-e'\) at \(e'\).
Additional example:

Uniform charge density sphere:

\[ \sigma_q = \frac{Q}{\left(\frac{4}{3}\pi R^3\right)} \]

\[ \mathbf{E}(r) = \frac{kQ}{r^2} \quad r \to R \]

For \( r < R \), \( \Phi_{\text{enc}} = \frac{4\pi}{3} r^3 \sigma_q = \Phi \left( \frac{r}{R} \right)^3 \)

\[ \mathbf{E}(r) = k\Phi \left( \frac{r}{R} \right)^3 \frac{1}{r^2} \quad \text{Gauss's law} \]