Lecture #2

I. Cross Sections & Decay Rates

Primary observables in nuclear & particle physics

1. Decay rate $\Gamma = \text{constant prob./time}$
   
   $$dN = -\Gamma N dt$$
   
   $\langle t \rangle = \frac{1}{\Gamma} \equiv \tau \quad \text{"lifetime"}$

2. Cross section: flux normalized scattering rate

   Classical: particles follow trajectories.

   Scattering angle $\Theta$ depends on impact parameter $b$

   $$dN = \frac{\sigma}{\sin \theta} d\Omega$$

   $$\text{flux } F \equiv \# \text{ incident particles } / \text{area/ster}$$

   $\phi \equiv \text{azimuthal angle}$

   All particles: then $bdbd\phi \to dN$
\[ \Delta N = \left( \text{rate} @ \theta, \varphi \right) \cdot \Delta \Omega = \frac{F \cdot b \Delta \theta \Delta \varphi}{F} \]

\[ \frac{\Delta \sigma}{\Delta \Omega} = \frac{b \Delta \theta \Delta \varphi}{\sin \theta \Delta \theta \Delta \varphi} = \frac{b}{\sin \theta} \frac{\Delta \varphi}{\Delta \theta} \]

\[ \frac{d\sigma}{d\Omega} = \frac{b}{\sin \theta} \left| \frac{\Delta \varphi}{\Delta \theta} \right| \]

(3) Non-Relativistic Quantum Scattering

No trajectory, amplitude to scatter. Asymptotic states are free particles (plane wave)

\[ \psi_i = \frac{1}{\sqrt{V}} \cdot e^{i \frac{P \cdot x}{\hbar}} \]

\[ \psi_f = \psi_i \cdot e^{-i \int V(r) \cdot d\tau} \]

\[ k^2 = k_f^2 \]

Amplitude \[ M_f i \]
\[ d\tau = \frac{dR}{F} \]

Q.m. flux includes incident wave function normalization of particle \( \psi \) per volume.

\[ F = \frac{1}{\sqrt{V}} \frac{\nu \Delta t}{a \Delta t} = \frac{\nu}{\sqrt{V}} \]

\( \nu \) incident particle speed.

\( (\text{Fermi's golden rule}) \quad \text{transition rate} \quad dR, \]

\[ dR = \frac{2\pi}{\hbar^2} |M_{fi}|^2 S(E) \]

Amplitude, dynamic \( \rho \frac{E}{m} \) density of states,kinetic

\[ S(E) = \frac{dN}{dE} = \frac{V}{(2\pi \hbar)^3} \int p^2 dp \frac{dR}{dE} = \frac{V}{(2\pi \hbar)^3} \frac{p^2}{dp} dE/dp \]

\[ \frac{dE}{dp} = \frac{d}{dp} \left( \frac{p^2}{2m} \right) = \frac{p}{m} = \nu \]

Note: relativistic case is the same,

\[ \frac{dE}{dp} = \frac{d}{dp} \left( \sqrt{p^2 + m^2} \right) = \frac{p}{E} = \frac{m \nu}{m^2} = \nu \]
Scattering Amplitude

In first order perturbation theory:

\[ M^{(0)}_{fi} = \int \frac{\psi^*}{f} V(x^3) \psi_i d^3x \]

\[ = \frac{1}{V} \int e^{i \frac{q \cdot x}{\hbar}} V(x^3) d^3x \]

where \( \vec{q} = \vec{k} - \vec{p} \) is momentum transfer.

\( M \) is just Fourier Transform of potential

For \( V(x^3) = \frac{ZZ' e^2}{|x^3|} \)

\[ M^{(0)}_{fi} = \frac{ZZ' e^2}{V} \int e^{i \frac{\vec{q} \cdot \vec{x}}{\hbar}} \frac{d^3x}{|x^3|} = \frac{ZZ' e^2}{V} \frac{4\pi \hbar^2}{|\vec{q}|^2} \]

Choose spherical coordinates, with \( \vec{q} \) as \( z \)-axis for integration.

Diagram representation ( Feynman-like )

\( E, \vec{k} \)

\( E', \vec{p} \)

\( \vec{q} = \vec{k} - \vec{p} \propto \frac{1}{|\vec{q}|^2} \)

Note: relativistically \( \vec{q} = \gamma (\vec{p}) \)

\[ \vec{q} \cdot \vec{q} = E \neq 0 \]

exchange of virtual photon "off mass shell"
Rutherford Cross Section

$$\frac{d\sigma}{d\Omega} = \left(\frac{\lambda}{\nu}\right) \frac{2\pi}{h} \left(\frac{2z'e^2}{\lambda}\right)^2 \left(\frac{2\pi}{h^2}\right)^2 \frac{V}{\lambda^4} \frac{p^2}{\nu} d\Omega$$

$$= 4 \left(\frac{p^2}{\nu^2}\right) \frac{1}{\lambda^4} (2z'e^2)^2$$

nm-relativistic \quad \frac{p}{\nu} = m \quad \therefore E = \frac{p^2}{2m}

$$\lambda^2 = \left(\frac{p}{\nu}\right)^2 = 2p^2(1 - \cos \Theta) = 4p^2 \sin^2 \Theta$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \left(\frac{m}{p^2}\right)^2 \left(\frac{2z'e^2}{\lambda^4}\right) \sin^4 \Theta \frac{1}{\sin^4 \Theta}$$

$$= \frac{1}{(4E^2)} \left(\frac{z'e^2}{\sin^4 \Theta}\right)$$

No \( \lambda \) \Rightarrow same as classical result.

Relativistic Cross Section

Relativistic scattering cross section is similar, written in terms of invariant amplitude \( M_{\gamma i} \):

$$1 + 2 \to 3 \ldots n + 2 \quad (n \text{ final particles})$$

$$d\sigma = \frac{1}{2E_1 2E_2 (U_1 + U_2)} \left| M_{\gamma i} \right|^2 d\Omega$$

invariant plane space

invariant flux
note: do being area 1 beam is lorentz invariant.

where invariant flux:

$$\overline{r}_i = E_i \left(1, \overline{v}_i^2 \right)$$

$$\overline{r}_2 = E_2 \left(1, \overline{v}_2^2 \right)$$

for $$\overline{v}_1 \cdot \overline{v}_2 = -v_1 v_2$$,

$$\left(\overline{v}_1 + \overline{v}_2\right)^2 = \left[v_1^2 + v_2^2 + 2v_1 v_2 \right]^{1/2}$$

$$= v_1 + v_2$$

$$2E_1 E_2 \left(v_1 + v_2\right) = 4 \sqrt{\left(\overline{r}_1 \cdot \overline{r}_2\right)^2 - m_1^2 m_2^2}$$

invariant phase space:

$$\Phi_n = \frac{\sqrt{d^3 P_i \left(\overline{r}_1 + \overline{r}_2 - \sum_{i=1}^{n} P_i \right)}}{2E_i (2\pi)^3}$$

Similarly, decay of particle w7 mass M → n particles in rest frame

$$d\Gamma = \frac{1}{2m_1} \left| M \right|^2 d\Phi_n$$

formulae are nice looking.

after lots of messaging and money

amount of factor of E, 2\pi.

note: $$\frac{d^3 P}{E}$$ is invariant.
The Rutherford Experiment

Thin foil scattering:

\[ \frac{\alpha}{\gamma} \frac{dN_\alpha}{dr} \stackrel{=} \frac{dN_\gamma}{dr} \]

Source

Collimator hole size area A

Foil, thickness l

"thin" means ignore multiple scattering

\[ F = \frac{dN_\alpha}{dt} \left( \frac{1}{A} \right) \times (\# \text{ targets}) \]

(\# of \( \alpha \)'s/time) thin hole

\[ \# \text{ targets} = \frac{N_0 S}{M_\alpha} (CA) \]

\[ \frac{\#}{\text{vol}} \]

\( N_0 = \) Avogadro's number

\( M_\alpha = A \times 10^{-3} \frac{\text{kg}}{\text{mole}} = A \times 1 \frac{\text{gm}}{\text{mole}} \) molarm mass

\( S = \) density \( \frac{\text{gm}}{\text{vol}} \)

\[ F = \frac{dN_\alpha}{dt} \left( \frac{N_0 S}{M_\alpha} \right) l \]
Measure cross section from observed scattering rate \( \frac{dn_s}{dt} \):

\[
\frac{d\sigma_{\exp}}{d\Omega} (\theta) = \frac{1}{\frac{d N}{d\Omega}} \frac{dN_s(\theta)}{d\Omega} (\theta) = \left( \frac{m_A}{4\pi \varepsilon} \right) \left( \frac{\frac{dN_s(\theta)}{d\Omega}}{\frac{dN_s}{d\Omega}} \right)
\]

Agreement with theoretical Rutherford formula verifies theoretical assumptions:

1. Mass assumed infinite
   \( M_N \gg M_{\text{Atom}} \)

2. Size is "point-like" \( \ll 0.1 \text{ nm atomic scale} \)

"Point-like" means follows \( \frac{d\sigma}{d\Omega} \propto \frac{1}{\sin^2 \theta} \bigg|_{\theta = \pi} = 1 \)

Extended object \( \theta \) would be expected to decrease faster w/ \( \theta \).

```
8 NUCLEAR SIZE AND NUCLEAR SHAPES

Fig. 2-1 The broad dashed curve gives the Coulomb cross section, and the solid curve represents the experimental data of Farwell and Wegner for Au, Pb, and Th. For Au, the finer theoretical curve corresponds to $R = 10.58 \times 10^{-13}$ cm and the coarser curve to $R = 10.3 \times 10^{-13}$ cm. For Pb, the finer curve corresponds to $R = 10.87 \times 10^{-13}$ cm and the coarser to $R = 10.42 \times 10^{-13}$ cm. For Th, the dashed curve corresponds to $R = 11.01 \times 10^{-13}$ cm (Eisberg and Porter).

--- Rutherford Scattering ---
Distance scale probed by $\alpha$:

Classical minimum distance $r_{\text{min}}$:

$$E_\alpha = \frac{\gamma_2 e^2}{r_{\text{min}}}$$

$$\frac{e^2}{h c} = \alpha = \frac{1}{137}$$ \text{ fine structure constant}

Take $E_\alpha = 4 \text{ MeV}$ on $^{207}_{82} \text{ Pb}$

$$r_{\text{min}} = \frac{2(82)}{137} \frac{200 \text{ MeV}}{4 \text{ MeV}} = \frac{8200}{137} \text{ fm}$$

$$= 60 \text{ fm}$$

Nucleon size $r_n \approx 1.2 \text{ fm} A^{1/3}$

$$= 1.2 (207)^{1/3} \text{ fm} = 7 \text{ fm}$$

$$r_{\text{min}} \gg r_{\text{pb}}$$

For $E = 25 \text{ MeV}$, $r_{\text{min}} \approx 10 \text{ fm}$, begin to probe nucleon size.
Low energy elastic $\pi$ - nulear scattering $\frac{d\sigma}{d\Omega}(\theta)$ shows structure that can be interpreted as wave scattering by absorptive sphere:

\[ \Delta = 2R \theta_n = n \lambda = n \frac{h}{P} \]

Note: first dip for $\pi$ - Pb scattering is slightly above 50°.
Mott cross section

Relativistic \((\nu = 1)\) \(e^-\) elastic scattering

by spinless, point-like nucleus of mass \(M\), charge \(Ze\):

\[
\frac{d\sigma^{\text{Mott}}}{d\Omega} = \frac{(E - hc)^2}{4E^2 \sin^2 \frac{\theta}{2}} \left( \frac{E}{E_0} \right) \sin^2 \frac{\theta}{2} \quad \text{ recoil} \quad \text{e- spin} \quad \text{e- spin}
\]

\(e^2 = p^2\)

Ultra-relativistic \(\nu = 1\)

In ultra-relativistic limit, helicity is conserved:

\[
\hbar = \left\langle \frac{\vec{S} \cdot \vec{p}}{|\vec{S}| / |\vec{p}|} \right\rangle = \pm 1
\]

\[
\frac{\vec{p} \to \vec{p}}{\vec{S} \to \vec{S}} \quad \text{"right handed"}
\]

\[
\frac{\vec{S} \to \vec{S}}{\vec{p} \to -\vec{p}} \quad \text{"left handed"}
\]

Incoming wave function with \(\vec{p}^2 = p^2\) has \(L_z = 0\). Conservation of angular momentum requires that \(\langle \vec{S} \cdot \vec{E} \rangle\) does not change.
\[ \langle \frac{1}{2}, \frac{1}{2} | \frac{1}{2}, \frac{1}{2} \rangle_p = \cos \frac{\theta}{2} \]

Take nuclear recoil into account

\[ \overrightarrow{P} = P_0 (1, \hat{z}) \quad \overrightarrow{p} = P (1, \hat{e}) \quad \overrightarrow{P} \cdot \overrightarrow{P}_0 = 0 \quad \overrightarrow{P} \cdot \overrightarrow{p} = 0 \]

4 momentum transfer:

\[ -q^2 = - (\overrightarrow{P}_0 - \overrightarrow{p})^2 = 2 \overrightarrow{P}_0 \overrightarrow{p} (1 - \cos \theta) \]

\[ = 4 E_0 E \sin^2 \frac{\theta}{2} \]

Energy conservation determines \( E(\theta) \):

\[ E_0 + m_N = E + E_N \]
\[(E_0 + m_N - E_i)^2 = E_N^2 = m_N^2 + p_N^2\]

\[|p_N|^2 = |p_o - p|^2 = p_o^2 + p^2 - 2p_o p \cos \theta\]

\[= E_o^2 + E^2 - 2E_o E \cos \theta\]

\[= (E_0 - E)^2 + 4E_0 E \sin^2 \theta \]

\[\sqrt{(E_0 - E)^2 + 2m_N (E_0 - E)} = (E_0 - E)^2 + 4E_0 E \sin^2 \theta \]

\[\left(\frac{1}{E} - \frac{1}{E_0}\right) = \frac{2 \sin^2 \theta}{m_N}\]

\[E^{-1} = E_0^{-1} \left(1 + \frac{2E_0}{m_N} \sin^2 \frac{\theta}{2}\right)\]
Nuclear Form factor:

\[
\mathcal{M}_f^\text{ud} = \int \psi_f^* \psi_i \, d^3x
\]

\[
\nabla^2 \phi(x^2) = -4\pi ze^2 \phi(x^2) \quad ; \quad \int_0^\infty \phi(x^2) \, d^3x = 1
\]

Use Green's theorem: \( \int (u \nabla^2 v - v \nabla^2 u) \, d^3x = 0 \)

to get

\[
\mathcal{M}_f^\text{ud} = \frac{-4\pi ze^2}{\nabla^2} \frac{\hbar^2}{g^2} \int e^{i \frac{q}{\hbar} \cdot \frac{x^2}{2}} \phi(x) \, d^3x
\]

nuclear form factor \( \equiv F(q^2) \)

as Feynman-like graph, point charge \( \rightarrow F(q^2) \)

\[
\frac{d\sigma}{d\Omega} = \frac{d\sigma^\text{tot}}{d\Omega} |F(q^2)|^2
\]
For \( \mathcal{P}(x) = f_0 e^{-x/a} \)

\[ F(q^2) = \left( 1 + \frac{q^2}{\alpha^2} \right)^{-1} \]

"dipole" better approximation,

\[ f(x) = f_0 \left[ 1 + \exp \left( \frac{x-a}{\alpha} \right) \right]^{-1} \]

data given \( c = 1.07 \) for \( A^{1/3} \); \( a = 0.54 \) fm

see figure on next page.

Homogeneous charged sphere approximation,

\[ R_{rms} = \sqrt{\langle r^2 \rangle} = 1.2 \text{ fm} \] for \( A^{1/3} \)
5 Geometric Shapes of Nuclei

- Nuclei are not spheres with a sharply defined surface. In their interior, the charge density is nearly constant. At the surface the charge density falls off over a relatively large range. The radial charge distribution can be described to good approximation by a Fermi function with two parameters

\[
\rho(r) = \frac{\rho(0)}{1 + e^{(r-c)/a}} .
\]  

(5.52)

This is shown in Fig. 5.8 for different nuclei.
- The constant \( c \) is the radius at which \( \rho(r) \) has decreased by one half. Empirically, for larger nuclei, \( c \) and \( a \) are measured to be:

\[
c = 1.07 \text{ fm} \cdot A^{1/3}, \quad a = 0.54 \text{ fm} .
\]  

(5.53)

- From this charge density, the mean square radius can be calculated. Approximately, for medium and heavy nuclei:

\[
\langle r^2 \rangle^{1/2} = r_0 \cdot A^{1/3} \quad \text{where} \quad r_0 = 0.94 \text{ fm} .
\]  

(5.54)

The nucleus is often approximated by a homogeneously charged sphere. The radius \( R \) of this sphere is then quoted as the nuclear radius. The following connection exists between this radius and the mean square radius:

\[
R^2 = \frac{5}{3} \langle r^2 \rangle .
\]  

(5.55)

Quantitatively we have:

\[
R = 1.21 \cdot A^{1/3} \text{ fm} .
\]  

(5.56)

This definition of the radius is used in the mass formula (2.8).
- The surface thickness \( t \) is defined as the thickness of the layer over which the charge density drops from 90% to 10% of its maximal value:

\[
t = r_{(\rho/\rho_0=0.1)} - r_{(\rho/\rho_0=0.9)} .
\]  

(5.57)

Its value is roughly the same for all heavy nuclei, namely:

\[
t = 2a \cdot \ln 9 \approx 2.40 \text{ fm} .
\]  

(5.58)