Due April 29, 2005

Physics 450/II Homework # 2

#1) The weak decay of the charged pion is almost completely to muon plus neutrino ($\pi^+ \to \mu^+\nu_\mu$, $\pi^- \to \mu^-\bar{\nu_\mu}$) and almost never to electron plus neutrino. (This is the primary source of cosmic ray muons.) Calculate the "helicity suppression factor" for the ratio of decay rates, using the fact that the probability for a left-chiral fermion to have right-helicity (and vice-versa) is given by $1 - v$, where the v is the particle speed $v = p/E$.

The next three problems concern the Dirac equation. The γ-matricies are defined by:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

In the Dirac representation they are represented explicitly by the Pauli matricies and the 2x2 unit matrix I:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$$(\gamma^0)^\dagger = \gamma^0, \ (\vec{\gamma})^\dagger = -\vec{\gamma}$$

,

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \ (\gamma^5)^\dagger = \gamma^5, \ \left\{\gamma^5, \gamma^\mu\right\} = 0$$

#2) From the Dirac equation $(i\partial_\mu\gamma^\mu - m)\psi = 0$ show that the 4-vector $j^\mu = e\bar{\psi}\gamma^\mu\psi$ satisfies the continuity equation, $\partial_\mu j^\mu = 0$. ($\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$) Remember that the Dirac conjugate is $\bar{\psi} = \psi^\dagger\gamma^0$ and the conjugate equation is $i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0$

#3) Show that the 4-current can be written in terms of left and right chiral components:

$$j^\mu = e\bar{\psi}\gamma^\mu\psi = e\bar{\psi}_L\gamma^\mu\psi_L + e\bar{\psi}_R\gamma^\mu\psi_R,$$

where $\psi_L = \frac{1}{2}(1 - \gamma^5)\psi$, $\psi_R = \frac{1}{2}(1 + \gamma^5)\psi$.
On the other hand show that the combination that gives mass to the fermion mixes L,R components:

$$m\bar{\psi}\psi = m\bar{\psi}_L\psi_R + m\bar{\psi}_R\psi_L$$

#4) A challenging (and interesting) problem from Griffiths: The explicit form of the Lorentz transformation of a Dirac spinor under a boost in the x-direction is given by the 4x4 matrix $S: \ \psi(x) \to \psi'(x') = S\psi(x')$

$$S = \begin{pmatrix} a_+ I & a_-\sigma_x \\ a_-\sigma_x & a_+ I \end{pmatrix}$$

where I is the 2x2 unit matrix, σ_x is the Pauli matrix, and $a_\pm = \pm\sqrt{(\gamma \pm 1)/2}$ where here $\gamma = 1/\sqrt{(1-v^2)}$ is the Lorentz transformation γ, **not a matrix!**

a) Show that

$$S^\dagger S = \gamma \begin{pmatrix} 1 & -v\sigma_x \\ -v\sigma_x & 1 \end{pmatrix} \neq I$$

and therefore $\psi^\dagger\psi$ is not a Lorentz invariant.

b) Show that $\bar{\psi}\psi$ is Lorentz invariant.

c) Show that $j^\mu = e\bar{\psi}\gamma^\mu\psi$ transforms as a 4-vector.