Worksheet on Addition of Angular Momentum
Due March 6, 2006

Let us write a general angular momentum eigenstate as $\chi_{j,m_j}$. This state is an eigenstate of the angular momentum operators $\hat{J}^2$, $\hat{J}_z$:

$$\hat{J}^2 \chi_{j,m_j} = \hbar^2 j(j+1) \chi_{j,m_j}$$
$$\hat{J}_z \chi_{j,m_j} = \hbar m_j \chi_{j,m_j}$$

where $j$ is an integer or half-integer, and $m_j$ takes on values,

$$m_j = -j, -j+1, ..., j-1, j.$$ 

Thus, $m_j$ is integer or half-integer according to whether $j$ is integral or half-integral. (note 1)

For example the $\chi_{j,m_j}$ can be the functions $Y_{l,m}(\theta, \phi)$ and the differential operators $L_i$, or the “spinor” wave functions $\chi^{1/2}_{l,m}$ and a set of $2 \times 2$ matrices $\hat{S}_i$ ($i = x, y, z$). We write the total angular momentum wave functions as the product states:

$$\phi_{l,m,s,m_s} = Y_{l,m} \chi^{1/2}_{l,m}.$$ 

The total angular momentum operator is the addition of the operators $\hat{L}$ and $\hat{S}$: $\hat{J}_i = \hat{L}_i + \hat{S}_i$ ($i = x, y, z$), which we can write as $\hat{J} = \hat{L} + \hat{S}$. (note 2) Do not worry that we have added a differential (“apple”) operator to a matrix (“orange”) operator, as each operator only acts on the corresponding wave function. For this reason, the states $\phi_{l,m,s,m_s}$ are eigenstates of $\hat{J}_z$:

$$\hat{J}_z \phi_{l,m,s,m_s} = \hat{L}_z Y_{l,m} \chi^{1/2}_{l,m} + Y_{l,m} \hat{S}_z \chi^{1/2}_{l,m} = \hbar (m_{\ell} + m_s) Y_{l,m} \chi^{1/2}_{l,m}.$$ 

The situation for the eigenstates of $\hat{J}^2$ is more complicated. This is because the states $Y_{l,m}$ are not eigenstates of $\hat{L}_x, \hat{L}_y$, and the states $\chi^{1/2}_{l,m}$ are not eigenstates of $\hat{S}_x, \hat{S}_y$. To see the complication, let’s write out the operator $\hat{J}^2$:

$$\hat{J}^2 = (\hat{L} + \hat{S}) \cdot (\hat{L} + \hat{S})$$
$$= \hat{L}^2 + \hat{S}^2 + 2 \hat{L} \cdot \hat{S}$$
$$= \hat{L}^2 + \hat{S}^2 + 2(\hat{L}_z \hat{S}_x + \hat{L}_y \hat{S}_y + \hat{L}_z \hat{S}_z)$$

1These bizarre relations and the rules that follow can be truly understood only from the application of group theory to the group SU(2).
2This is because the eigenvalues transform as components of a Euclidean vector.
What are the eigenstates of \( \hat{J}^2 \), \( \hat{J}_z \) that we denote by \( \chi_{j,m} \)? They are known, linear combinations of \( Y_{l,m_l}\chi_{\frac{s}{2},m_s} \), given in terms of coefficients (known as Clebsch-Gordon coefficients) that are found in tables. I will denote these numbers as \( C_{m_l,m_s} \), but keep in mind that there are different sets of constants for each \( l, s \) and corresponding \( j \).

\[
\chi_{j,m} = \sum C_{m_l,m_s} Y_{l,m_l}\chi_{\frac{s}{2},m_s},
\]

where the sum runs over all combinations of \( m_l,m_s \) such that \( m_j = m_l + m_s \).

Note that this is required for these combinations to be eigenstates of \( \hat{J}_z \).

More generally,

\[
\chi_{j,m_j} = \sum C_{m_1,m_2}\chi_{j_1,m_1}\chi_{j_2,m_2},
\]

were the sum runs over combinations \( m_j = m_{j_1} + m_{j_2} \).

What is the corresponding \( j \)? There are actually a set of them, given by the rule:

\[
|j_1 - j_2| < j < j_1 + j_2,
\]

where \( j \) takes on all values in unit steps between these limits:

\[
j = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2 - 1, j_1 + j_2
\]

As long as the system conserves total angular momentum, the quantum numbers \( j, m_j \) will label the energy eigenstates, and the states of given \( j \) will form a multiplet of degenerate states of dimension \( d_j \) (number of states)

\[
d_j = 2j + 1
\]

Each of the \( d_j \) states has a different value of \( m_j \). The total number of states must stay the same, so that the dimensions follow the rule:

\[
d_{j_1} \times d_{j_2} = \sum d_j
\]

In class I showed how to add the angular momentum of two spin-\( \frac{1}{2} \) wave functions. The spins add to form two multiplets, a dimension one (singlet) and a dimension three (triplet) with corresponding \( j = 0, 1 \). The total numbers of states add up as \( 2 \times 2 = 1 + 3 \).
As an example, I will write down the explicit linear combinations for the $P_{1/2}$ and $P_{3/2}$ multiplets. For $P_{3/2}$:

\begin{align*}
\chi_{3/2,3/2} &= Y_{1,1}\chi_{1/2,1/2} \\
\chi_{3/2,1/2} &= \sqrt{1/3}Y_{1,1}\chi_{1/2,-1/2} + \sqrt{2/3}Y_{1,0}\chi_{1/2,1/2} \\
\chi_{3/2,-1/2} &= \sqrt{1/3}Y_{1,-1}\chi_{1/2,1/2} + \sqrt{2/3}Y_{1,0}\chi_{1/2,-1/2} \\
\chi_{3/2,-3/2} &= Y_{1,-1}\chi_{1/2,-1/2}
\end{align*}

And for $P_{1/2}$:

\begin{align*}
\chi_{1/2,1/2} &= \sqrt{2/3}Y_{1,1}\chi_{1/2,-1/2} - \sqrt{1/3}Y_{1,0}\chi_{1/2,1/2} \\
\chi_{1/2,-1/2} &= -\sqrt{2/3}Y_{1,-1}\chi_{1/2,1/2} + \sqrt{1/3}Y_{1,0}\chi_{1/2,-1/2}
\end{align*}

Notice that in each case, $m_j = m_\ell + m_s$, that the states remain “orthogonal” and properly normalized. The full H-atom wave functions would be multiplied by the appropriate $R_{n,\ell}$.

Here we will not be concerned with the linear combinations, but will work through the rules for finding the $j$’s and showing the dimension rule holds.
**exercise 1:**

Find values of $j$ for the following case, and show that the dimension rule holds in each case:

a) $j_1 = 0, j_2 = 1/2$ (for atomic states, these are denoted by $nS_j$)

b) $j_1 = 1, j_2 = 1/2$ (for atomic states, these are denoted by $nP_j$)

c) $j_1 = 2, j_2 = 1/2$ (for atomic states, these are denoted by $nD_j$)

d) $j_1 = 1, j_2 = 1$

**exercise 2:**

We add a term to the energy operator for the hydrogen atom to take into account for the spin-orbit ("so") coupling:

$$
\hat{H}_{so} = E_{so} \frac{\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}}{\hbar^2}
$$

The states $\chi_{j,m_j}$ are eigenstates because

$$
\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} = \frac{1}{2}(\hat{\mathbf{J}}^2 - \hat{\mathbf{L}}^2 - \hat{\mathbf{S}}^2)
$$

Find the eigenvalues of $\hat{H}_{so}$ for the combinations a-c (but not d) from exercise 1.

**exercise 3:**

The hydrogen atom ground state is 4-fold degenerate when we include the proton spin in addition to the electron spin. However, there is an additional energy due to the interaction between the proton and electron magnetic moments: a spin-spin coupling ("ss"). To take this interaction into account, we add a term to the energy operator for the hydrogen atom:

$$
\hat{H}_{ss} = E_{ss} \frac{\hat{\mathbf{S}}^p \cdot \hat{\mathbf{S}}^e}{\hbar^2}
$$

This gives rise to the hyperfine splitting. Just as before, the states are linear combinations of the products $\chi_{n,m_s}^p \chi_{\pm m_s}^e$ and the spin operators act only on their respective spinor wave functions. The ground state has no orbital angular momentum, so the total angular momentum is just the sum of the two electron spins.

Find the eigenvalues of $\hat{H}_{ss}$ for the multiplets of total spin ($s = 1$ and $s = 0$) in analogy to exercise 2.