Lecture 9: Spin

Electron has "intrinsic" spin.

Classical current loop for magnetic moment:

\[ \mathbf{\mu} = I \mathbf{a} \]

Rewrite in terms of \( \mathbf{r} \):

\[ L = m r^2 \omega \]
\[ I = e \frac{V}{2 \pi r} = \frac{e \omega}{2 \pi} \]
\[ a = \frac{A r^2}{2} \]

\[ \mathbf{\mu} = \frac{e \omega}{2m} \mathbf{r} \]

For spinning sphere with uniform surface charge,

\[ \mathbf{\mu} = \frac{e}{2m} \left( \frac{5}{3} \right) \mathbf{r} \]

\[ J = \text{gyromagnetic ratio} \]

In magnetic field, torque is

\[ \mathbf{N} = \frac{d\mathbf{L}}{dt} = \mathbf{\mu} \times \mathbf{B} = \frac{e}{2m} \mathbf{r} \times \mathbf{B} = \mathbf{L} \times \mathbf{B} \]

\( \mathbf{L} \) vector precesses about \( \mathbf{B} \) field direction with

\[ \omega_L = \frac{e}{2m} \frac{\mathbf{r}}{B} \quad \text{Larmor frequency} \]
Potential energy of dipole:

\[ V = -\vec{\mu} \cdot \vec{B} \]

In a non-uniform \( \vec{B} \) field, dipole will experience a force.

Take \( \vec{B} = B(z) \hat{z} \)

\[ F_z = -\nabla V = -\mu_z \frac{dB}{dz} \]

Classically, \( \mu_z \) takes on continuous values, \( -\mu < \mu_z < \mu \)

Stern Gerlach experiment

Silver atoms have single \( e^- \) in outer shell:

Over \( \text{Collimated} \quad SG_{1/2} \)

Interpretation: \( S_z = \pm \frac{1}{2} \) quantized spin

2 beams!
We have a new (quantized) degree of freedom for the electron. We must modify the wave function as:

\[ \psi_e = \Psi_{\text{space}}(r) \chi(\text{spin}) \]

\[ \chi \text{ = spin state, does not depend on } r \text{ but can depend on } t. \]

Based on quantum rule for \( L \):

\[ -m_L < L < +m_L \] (\( 2L+1 \) states)

Two quantum states implies \( S = \frac{1}{2} \)

\[-m_S < S < m_S \] (\( 2S+1 \) states)

\[ m_S = \pm \frac{1}{2}, \quad S = \frac{1}{2} \]

Can be put on firmer theoretical ground from theory of symmetry (group theory) and properties of rotation, Lorentz groups.

All together, energy eigenstates are

\[ \Phi_{n,L,m_L,\frac{1}{2},m_S} = \text{Rae } Y_L^m \chi_{m_S} \]
Aside: "Spin-$\frac{1}{2}$" Particles

Although we will never need this, it is neat to know that $\chi_{\pm \frac{1}{2}}$ can be represented as 2-component complex objects

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} ; \quad |a|^2 + |b|^2 = 1$$

$$\chi_{+ \frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \quad \chi_{- \frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$a$, $b$ are amplitudes to have spin component $\pm \hbar/2$.

Spin operators are 2x2 (complex) matrices.

When you add special relativity to Q.M., Dirac equation automatically gives e-intrinsic spin and predicts

$$g_e = 2(1 + g_s^2)$$

where $g_s^2 \approx 10^{-3}$ instead of $\frac{1}{10}$ precise.

elementary, point-like particle

$$\vec{p}_e = \frac{e}{2m} \vec{S} = \frac{e}{m} \vec{S}$$

$$\vec{p}_e \cdot \vec{B} = \pm \left( \frac{eB}{2m} \right) B_2$$

Bohr Magneton $= 5.78 \times 10^{-5} \text{eV/T}$
proton, neutron also have spin \( \frac{1}{2} \):

\[
\mu' = \left( \frac{g \hbar}{2m} \right) \left( \frac{e}{2m} \right)
\]

nuclear magneton \( 3.15 \times 10^{-8} \text{ ev/}\tau \)

Because of mass, down by \( \sim 10^{-4} \)

\[
\frac{g \hbar}{2} = 2.793 \quad \frac{g \nu}{2} = -1.913
\]

p, n not elementary particles!

Relativistic correction to H-atom

Small: \( \langle v \rangle = \Delta c \)

Energy shift \( \Delta E \propto \Delta^2 E \)

But measurable!

Predicted by Dirac equation, collectively referred to as "Fine structure."
Free Structure

1. Relativistic Kinetic Energy
2. Spin-orbit
3. Darwin (electron field, Zitterbewegung), $h/mc = 2.43 \times 10^{-8} \text{nm}$.

Spin-orbit modified eigenstates

Physical picture: orbiting e-spin magnetic moment coupled to internal $B$ induced by relative motion of proton (+ relativistic effect known as Thomas precession).

- $L, S$ not separately conserved
- $J^z = L^z + S^z$ total angular momentum must be conserved

$\Delta E_{so} \propto \left\langle L^z S^z \right\rangle = \frac{1}{2} \left\langle J^z - L^z - S^z \right\rangle$

$= \frac{\hbar^2}{2} \left( 2J(J+1) - l(l+1) - \frac{3}{4} \right)$

"Good" quantum number (conserved)
label eigenstates:

$\nu, j, m_j, l, s$
my all fine same energy, so form (degenerate) multiplets \( n L j \) where in spectroscopic notation \( L = 0, 1, 2, 3, \ldots \)
\[ = s, p, d, f, \ldots \]

Rule for determining \( j \): (group theory)
\[ |l-s| < j < l+s \]

Example \( n = 2 \):

\( l=0, s=\frac{1}{2} \Rightarrow j = \frac{1}{2}, 2s+\frac{1}{2} \)
\( l=1, s=\frac{1}{2} \Rightarrow j = \frac{3}{2}, \frac{1}{2}, 2p+\frac{1}{2}, 2p_{\frac{1}{2}} \)

# of states with same \( \ell \) spin

Original \( 2(2\ell+1) = 6 \) degenerate states "broken" into \( \ell = 2j+1 \) multiplets

In general: \( (2j+1) \times (2j' + 1) = \sum_{j=|j-j'|}^{j+j'} (2j+1) \)

Total fine structure:

Including all fine structure effects (Dive equa) \( 2s_{\frac{1}{2}}, 2p_{\frac{1}{2}} \) are degenerate
Total Fine Structure:

\[ \Delta E_{n'1} = -\alpha^2 E_n \left(3 - \frac{8n}{2j' + 1}\right) \]

\[ n = 2 : \quad \frac{\alpha^2}{4} \frac{1}{2} \left( \frac{mc^2 \alpha^2}{\lambda} \right) \left(3 - \frac{16}{2j' + 1}\right) \]

\[ = \frac{mc^2 \alpha^4}{2^7} \left(3 - \frac{16}{2j' + 1}\right) \]

\[ = \frac{mc^2 \alpha^4}{2^7} \left[3 - 1\right] \]

\[ \approx 1.13 \times 10^{-5} \text{ eV} \]

Splitting:

\[ \Delta E_{fs} = 4 \frac{mc^2 \alpha^4}{2^7} = 4 \times 10^{-6} \text{ eV} \]

But \( 2S_{1/2}, 2P_{1/2} \) are not degenerate, but are split by \underline{Lamb shift} \( 1057 \text{ MHz} \)

A quantum field theoretic effect, accurately calculated in QED. (Includes vacuum polarization)

\[ \Delta E (2S_{1/2} - 2P_{3/2}) = 1057.86(2) \text{ MHz} \]

\[ \Delta E_{\text{exp}} = 1057.845(9) \text{ MHz} \]
Structure of Hydrogen

Fig. 4.2 Low-lying energy levels of atomic hydrogen. The diagram is not drawn to scale.

From Bjorken & Drell, Relativistic Quantum Mechanics

Not drawn to scale.
Atomic transition:

\[ \text{Spontaneous emission} \quad \tau = 10^{-8} \text{ s} \quad (\text{"allowed"}) \]

also "supernova" \[ \tau = 10^{-5} \text{ s}, \text{ "metastable" (2\gamma)} \quad \gamma = \frac{1}{25} \]

\[ \begin{array}{c}
\downarrow \\
1S_\frac{1}{2} \\
2S_\frac{1}{2}
\end{array} \quad \rightarrow \quad \frac{\hbar c}{\lambda} = E_2 - E_1 = 10.2 \text{ eV} \]

\[ \lambda = \frac{\hbar c}{10.2 \text{ eV}} = \frac{1240 \text{ eV nm}}{10.2 \text{ eV}} = 122 \text{ nm} \]

Energy eigenstates make transitions as a result of EM vacuum fluctuations.

Photon for spin 1. Allowed (electric dipole) selection rule

\[ \Delta \ell = \pm 1 \quad \text{conservation of angular momentum} \]

\[ \Delta J = \pm 1, 0 \quad \text{w/ spin–orbit coupling} \]

\[ J = 0 \rightarrow J = 0 \quad \text{forbidden} \]
**Zeeman Effect**

See spectral lines from individual $m_j$ by applying a weak external $\vec{B}$.

\[
\Delta E = -\mu \cdot \vec{B} = \frac{\alpha}{2m} (\vec{L} + \vec{S}) \cdot \vec{B}
\]

e.g. factor

**Weak field** $\Delta E \ll \Delta E_{fine}$ structure

Then $l, s, m_l, m_s$ multiplets are a good description. Total $J'$ precession about $\vec{B}$.

\[
\Delta E = \frac{e}{2m} \left[ \frac{(\vec{L} + \vec{S}) \cdot \vec{J}'}{J'^2} \right] \left[ \vec{J}' \cdot \vec{B} \right]
\]

\[
\vec{J}' \cdot \vec{B} = m_j' \cdot B
\]

\[
\Delta E = \mu_B m_j' B g_L
\]

when Lande-$g$ factor is $3/2$

\[
g_L = 1 + \frac{1}{2} \frac{j(j+1) + s(s+1) - l(l+1)}{j(j+1)}
\]

*Note: I have used $(\vec{L} + \vec{S}) \cdot \vec{J} = (\vec{J} + \vec{S}) \cdot \vec{S} = J^2 + S^2$

\[
= J^2 + \frac{J^2 + S^2 - L^2}{2}
\]
Example:

2S1/2: \( g_L = 1 + \frac{\frac{1}{2}(\frac{3}{2}) + \frac{1}{2}(\frac{3}{2})}{2(\frac{1}{2})(\frac{3}{2})} = 2 \)

\( \Delta E = g_L m_j \mu_B B = 2(\frac{1}{2}) \mu_B B = \pm 5.79 \times 10^{-5} \text{eV/} \mu \text{B} \)

so for \( B \sim 1 \text{ Tesla} \), \( \Delta E \sim \Delta E(\text{fine structure}) \)

"Weak field" \( B \ll 0.1 \text{ Tesla} \)

2P3/2: \( g_L = 1 + \frac{\frac{3}{2}(\frac{5}{2}) + \frac{1}{2}(\frac{3}{2}) - 1(\frac{1}{2})}{2(\frac{3}{2})(\frac{5}{2})} = \frac{2}{3} \)

2P1/2: \( g_L = 1 + \frac{\frac{1}{2}(\frac{3}{2}) + \frac{1}{2}(\frac{3}{2}) - 1(\frac{1}{2})}{2(\frac{1}{2})(\frac{3}{2})} = \frac{4}{3} \)

Strong field: \( B \gg 1 \text{ Tesla} \)

Ignore fine structure splitting:

\( \Delta E = (m_e + 2m_s) \mu_B B \)