Quantum Density of States:
Consider molecular rotation, \( E_e = \frac{h^2}{2I} \ell (\ell + 1) \)

Number of molecules with energy \( E_e \) at temp. \( T \):

\[
N(E_e, T) = N_0 (2\ell + 1) e^{-\frac{E_e}{k_B T}}
\]

\( \ell \) ~ statistical "density of states" 
\( N_0 \) ~ total number of molecules

Ratio of populations:

\[
\frac{N(E_e', T)}{N(E_e, T)} = \frac{2\ell' + 1}{2\ell + 1} e^{-\frac{E_e' - E_e}{k_B T}}
\]

Example: interstellar cyanogen molecule (CN)

\( \lambda \) scattered light

\[\xrightarrow{\text{observe}}\]

\( \text{cloud of gas} \)

\( \text{observe absorption: } \lambda_0, \lambda_1 \)

\( \lambda_0 = 387.5 \text{ nm}; \Delta \lambda = 0.061 \text{ nm} \)

\[
\Delta E = \frac{hc}{\lambda_0} - \frac{hc}{\lambda_1} \approx \frac{hc \Delta \lambda}{\lambda_0^2}
\]
\[ \Delta E = \frac{\hbar c \Delta \lambda}{\lambda_0^2} = \frac{(1240 \text{ ev} \cdot \text{nm}) (0.06 \text{ nm})}{(387.5 \text{ nm})^2} = 5 \times 10^{-3} \text{ eV} \]

\[ \frac{T_1}{T_0} = \frac{N_1}{N_0} = \frac{3}{1} \exp\left(-\frac{\Delta E}{k_B T}\right) \approx \frac{1}{4} \]

observed value

Then

\[ \frac{\Delta E}{k_B T} = \ln(12) \]

\[ T = \frac{\Delta E}{k_B \ln(12)} = \frac{5 \times 10^{-3} \text{ eV}}{\ln(12)} \left( \frac{300 \text{ K}}{0.026 \text{ eV}} \right) \]

\[ T \approx 2.3 \text{ K} \]

"Temperature of intergalactic space"

Note: this was done before the observation of the cosmic microwave background.
Density of state for photons:

To count modes, put EM wave in box with conducting walls; $\mathbf{E}_n$ at box wall is zero.

Standing wave $k_z = \frac{n\pi}{L}$

$$\Delta N = 2 \left( \frac{L}{\pi} \right) \Delta k_z$$

2 polarizations

In 3 dimensions, $\Delta N = 2 \left( \frac{L}{\pi} \right)^3 \Delta k_x \Delta k_y \Delta k_z$

In limit $L \to \infty$,

$$dn = \frac{dN}{V} = 2 \frac{d^3k}{(2\pi)^3} \quad k_x, k_y, k_z > 0$$

$$= 2 \frac{d^3k}{(2\pi)^3}$$

where now each $k_i$ goes from $-\infty$ to $\infty$.

Each standing wave corresponds to a superposition of left/right traveling waves, so we can think of this as counting photons as well.
Density of state for particles:

Since q.m. particles have de Broglie wavelength \( \lambda = \hbar/p \), the form is the same:

\[
\frac{dN}{d\Omega} \frac{d^3k}{(2\pi)^3} \quad p = \hbar k = \frac{\hbar}{\lambda}
\]

Spin multiplicity.

Note: photons have spin 1 (1 but only ±1 in spin orientation), \( \langle \hat{S}_z \rangle = \pm \frac{1}{2} \). This corresponds to two polarization states and is due to the photon having zero mass. (The longitudinal polarization state \( \langle \hat{S}_z \rangle = 0 \) is missing.)

Black Body Spectrum:

All objects emit EM radiation with characteristic spectrum, differential flux:

\[
\text{Flux} = \frac{dR}{df} \quad \text{energy emitted per unit time from } \lambda \text{ to } \lambda + d\lambda
\]

\[
\frac{dR}{d\lambda} = \left( \frac{dR}{df} \right) \frac{c}{\lambda^2}
\]
Spectrum is "thermal", obey

\[ R(T) = \frac{\varepsilon(T)}{\sigma} T^4 \]

\[ \sigma = 5.6 \times 10^{-8} \text{ W/m}^2/\text{K}^4 \]

\[ \text{Watts} = \text{Joules/sec.} \]

Efficiency factor

\[ \varepsilon(T) = \text{property of material} \quad \varepsilon \leq 1 \]

\[ \varepsilon(T) = 1 \quad \text{"perfect" black body} \]

Wien's displacement law:

\[ \lambda_{\text{peak}} T = 0.00290 \text{ m.K} \]

or

\[ \lambda_{\text{peak}} (k_B T) = 250 \text{ eV.nm} \]

Sun's spectrum (non-perfect black body):

\[ \lambda_{\text{peak}} = 500 \text{ nm} \]

\[ T = 5800 \text{ K} \]

from which you can calculate the total

radiated power from the Sun,

\[ P = \frac{4 \pi (R_5^2)}{5 \lambda_{\text{peak}}} \sigma T^4 \]

\[ R_5 = 7 \times 10^8 \text{ m} \]

\[ \approx = 4 \pi (4.3) \times 0.17 \text{ m}^2 \times 10^{-8} \frac{\text{W}}{\text{m}^2 \text{ K}^4} \times (5.8)^4 \times 10^4 \text{ K}^4 \]

\[ = 4 \pi (4.3) \times 5.6 (\text{ eV}) \times 10^{-8} + 3 \times 12 \text{ eV} \times 10 \]

\[ = 3.8 \times 10^8 \text{ W} \]
Perfect black body: radiation in box

\[
\frac{dU}{dK (KT)} = \frac{4kR}{dK} \quad \text{small hole}
\]

Cavity @ T

\[
U(T) = \text{total energy density of EM radiation in cavity, velocity factors}
\]

\[
\frac{dR}{dK} = \frac{c}{4} \left( \frac{dU}{dK} \right)
\]

\[
\text{geometric factor}
\]

Classically expect: 2 deg of freedom in classical oscillators

\[
\frac{dU}{dK} = \frac{dn}{dK} \quad \langle E_K \rangle = \frac{dn}{dK} \frac{1}{2k_BT} \]

\[
\text{number of modes } k, k+dK
\]

\[
n = \int \frac{2}{(2\pi)^3} d^3k = \frac{2 \cdot 4\pi}{8 \pi^3} k^2 dk = \frac{k^2}{\pi^2} dk
\]

\[
\frac{dU}{dK} = \frac{k^2}{\pi^2} (k_BT)
\]

\[
U(T) = (k_BT) \frac{1}{\pi^2} \int_0^\infty k^2 dk \to \infty \quad \text{ultraviolet divergence}
\]
Quantum Distribution for photons (photons)

\[ E_g = \hbar \omega \text{, so might expect that we should add a Boltzmann factor: } e^{-\frac{\hbar \omega}{k_B T}}. \]

Correct factor is:

\[ f_g = \frac{1}{e^{\frac{-\hbar \omega}{k_B T}} - 1} \]

\[ \rightarrow \quad e^{\frac{-\hbar \omega}{k_B T}} \]

\[ k \rightarrow \infty \]

\[ \rightarrow \quad \frac{1}{(1 + \frac{\hbar \omega}{k_B T})^{-1}} = \frac{k_B T}{\hbar \omega} \]

This will give correct classical limit of \( dn \propto \)

\[ k \rightarrow 0. \]

\[ \frac{dN}{dk} = \frac{k^2}{\pi^2} \left( \frac{1}{e^{\frac{\hbar \omega}{k_B T}} - 1} \right) \rightarrow \frac{k^2}{\pi^2} \frac{k_B T}{\hbar \omega} = \frac{k}{\pi^2 \hbar} \left( k_B T \right) \]

Underlying assumptions:

1) Identical Bose particles

2) No fixed photon number
Returning to \( \frac{dU}{dk} = \frac{dN}{dk} \cdot \text{hck} \) – energy of photon

\[
\frac{dU}{dk} = \frac{k^2}{\pi^2} \left( \frac{1}{e^{\frac{\pi c k}{k_b T}} - 1} \right) \cdot \text{hck}
\]

\( x = \frac{\text{hck}}{k_b T} \)

\[
dU = \left( \frac{1}{\text{hck}} \right)^3 \left( \frac{R_b T}{k_b T} \right)^4 \left( \frac{x^3 \, dx}{e^x - 1} \right)
\]

\[
\int_0^\infty x^3 \, dx = \frac{\pi^4}{15}
\]

\( U(T) = \left( \frac{1}{\text{hck}} \right)^3 \frac{\pi^4}{\pi^2} \frac{1}{15} \left( \frac{k_b T}{k_b} \right)^4 \)

\[
R(T) = \frac{c}{4} \cdot U(T) = \frac{(2\pi)^3}{k_b^4} \cdot \frac{\pi^2}{9 \cdot 15} \left( \frac{R_b T}{k_b} \right)^4
\]

\[
\sigma = \frac{2\pi^5 k_b^4}{15 \cdot h^3 c^2}
\]

Wien's law: \( \sigma^2 U/dk^2 = 0 \)

Classical limit:

\[
\frac{dU}{dk} \to \frac{k^2}{\pi^2} \cdot \frac{\text{hck}}{\text{hck} / k_b T} = \frac{k^2}{\pi^2} \left( \frac{R_b T}{k_b} \right)
\]
radiated power

\[
\frac{d\nu}{d^3k} \rightarrow R = \text{energy transported} \quad \text{area} \cdot \text{time}
\]
cavity @ T

photon density:

\[
\frac{d\nu}{d^3k} = \frac{1}{(2\pi)^3} f_x(k)
\]

where \( f_x(k) = \frac{1}{e^x - 1} \quad x = \frac{k \chi k}{k_B T}
\]
depends only on \( |\mathbf{k}| \)

\[\rightarrow \mathbf{A}\]

\[\mathbf{A} \cdot \mathbf{k} = (A \cos \theta) \cos \theta_k
\]

Energy through area \( A \) in time \( \Delta t \):

\[R \cdot A \cdot \Delta t = (A \Delta t \cos \theta) \int \cos \theta_k \frac{d\nu}{d^3k} \quad (2\pi)^3 \quad \text{rad}^2 \text{cm}^{-2} \text{s}^2 \text{~erg}^{-1}
\]
\[R = c \int_{0}^{\frac{\pi}{2}} k^2 dk (1 - k^2)(\frac{d\nu}{d^3k}) \int_{0}^{\frac{\pi}{2}} \sin \theta d\theta \int \cos \theta d\theta
\]
\[ R = c \int_0^\infty k^2 dk \left( \frac{d\nu}{d^3k} \right) \frac{1}{2\pi^2} \int_0^1 x dx \]

\[ R = \frac{c}{\pi} \int_0^\infty 4\pi k^2 dk \left( \frac{d\nu}{d^3k} \right) \]

\[ = \frac{c}{\pi} \int_0^\infty \left( \frac{d\nu}{d^3k} \right) k^2 d^3k = \frac{c}{\pi} U(T) \]

\[ R = \frac{c}{\pi} U(T) = \frac{\Omega_B}{\epsilon} T^4 = \frac{\sqrt{g}}{\epsilon} (R_B T)^4 \]